A Summary of Course Outlines for High School

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Overview of High School Course Outlines

Algebra 1	Geometry	Algebra 2
A0 Introduction	G0 Introduction and Construction	A0 Introduction
A1 Modeling With Functions	G1 Basic Definitions and Rigid Motion	A1 Exponential Functions
A2 Linear Functions	G2 Geometric Relationships and Properties	A2 Trigonometric Functions
A3 Linear Equations & Inequalities in One Variable	G3 Similarity	A3 Functions
Modeling Unit 1	M1 Geometric Modeling 1	Modeling Unit 1
A4 Linear Equations & Inequalities in Two Variables	G4 Coordinate Geometry	A4 Rational and Polynomial Expressions
A5 Quadratic Functions	G5 Circles and Conics	P1 Probability
A6 Quadratic Equations	G6 Geometric Measurements and Dimensions	S2 Statistics
S1 Statistics	G7 Trigonometric Ratios	Modeling Unit 2
Modeling Unit 2	M2 Geometric Modeling 2	

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Algebra 1 Course Outline

Content Area	Formative Assessment Lessons	# of Days
A0 Introduction: Mathematical Investigations		5
A1 Modeling with Functions N-RN The Real Number System 3 N-Q Quantities 1, 2, 3. F-IF Interpreting Functions 1, 2, 3, 4, 5, 9 F-LE Linear, Quadratic and Exponential Models 1, 2, 3, 5 ← F-IF 1,2,3,4,5 → ← N-Q 1,2,3; F-IF 9; F-LE 1,2,3,5 → ← N-RN 3 →	Functions and Everyday Situations Modeling: Having Kittens Rational and Irrational Numbers 1	15
A2 Linear Functions F-IF Interpreting Functions 6, 7a, 9 F-BF Building Functions 1a F-LE Linear, Quadratic and Exponential Models 1 ← F-IF 6 → ← F-IF 7a,9; F-BF 1a; F-LE 1 →	Comparing Investments Generalizing Patterns: Table Tiles	16
A3 Linear Equations & Inequalities in One Variable A-APR Arithmetic with Polynomials and Rational Expressions 1, 3 A-SSE Seeing Structure in Expressions 1, 2, 3a, 3b, 3c A-REI Reasoning with Equations and Inequalities 1, 3, 11 A-CED Creating Equations 1, 3, 4 ← A-SSE 1,2; A-APR 1; A-CED 1,3,4; A-REI 1,3,11 → ← A-SSE 3; A-APR 3 →	Manipulating Polynomials Interpreting Algebraic Expressions Sorting Equations and Identities Building and Solving Equations 2	16
A4 Linear Equations & Inequalities in Two Variables A-CED Creating Equations 2, 3, 4 A-REI Reasoning with Equations and Inequalities 5, 6, 10, 12 ← A-CED 2,3,4; A-REI 10,12 → ← A-REI 5,6 →	Solving Linear Equations in Two Variables Optimization Problems: Boomerangs Defining Regions Using Inequalities	16
A5 Quadratic Functions F-IF Interpreting Functions 4, 5, 6, 7a, 7b, 8a, 9 F-BF Building Functions 1a, 3 ← F-IF 4,5,6 → ← F-IF 7a,7b,8a,9; F-BF 1a → ← F-BF 3 →	Forming Quadratics	20
A6 Quadratic Equations A-SSE Seeing Structure in Expressions 3a, 3b A-REI Reasoning with Equations and Inequalities 4a, 4b ← A-REI 4a,4b → ← A-SSE 3a,3b →	Solving Quadratic Equations: Cutting Corners	26
S1 Statistics S-ID Interpreting Categorical and Quantitative Data 1, 2, 3, 4, 5, 6, 7, 8, 9 ← S-ID 7,8,9 → ← S-ID 5,6 → ← S-ID 1,2,3,4 →	Representing Data 1: Using Frequency Graphs Representing Data 2: Using Box Plots Interpreting Statistics: A Case of Muddying the Waters Devising a Measure for Correlation	30
Total days		144

Most of the Formative Assessment Lessons are available at http://map.mathshell.org/materials/lessons.php

Geometry Course Outline

Content Area	**Formative Assessment Lessons**	**# of Days**
G0 Introduction and Construction G-CO Congruence 12, 13 G-CO 12,13		12
G1 Basic Definitions and Rigid Motion G-CO Congruence 1, 2, 3, 4, 5, 6, 7, 8 G-CO 6,7,8 · G-CO 1,2,3,4,5	Representing and Combining Transformations Analyzing Congruency Proofs	20
G2 Geometric Relationships and Properties G-CO Congruence 9, 10, 11 G-C Circles 3 G-CO 9,10,11 · G-C 3	Applying Angle Theorems Evaluating Statements About Length and Area Inscribing and Circumscribing Right Triangles	15
G3 Similarity G-SRT Similarity, Right Triangles, and Trigonometry 1, 2, 3, 4, 5 G-SRT 1,2,3,4,5	Geometry Problems: Circles and Triangles Proofs of the Pythagorean Theorem	20
M1 Geometric Modeling 1 G-MG Modeling with Geometry 1, 2, 3 G-MG 1,2,3	Solving Geometry Problems: Floodlights	7
G4 Coordinate Geometry G-GPE Expressing Geometric Properties with Equations 4, 5, 6, 7 G-GPE 4,5,6,7	Finding Equations of Parallel and Perpendicular Lines	15
G5 Circles and Conics G-C Circles 1, 2, 5 G-GPE Expressing Geometric Properties with Equations 1 G-C 1,2,5 · G-GPE 1	Equations of Circles 1 Equations of Circles 2 Sectors of Circles	15
G6 Geometric Measurements and Dimensions G-GMD 1, 3, 4 G-GMD 1,3,4	Evaluating Statements About Enlargements (2D & 3D) 2D Representations of 3D Objects	15
G7 Trigonometric Ratios G-SRT Similarity, Right Triangles, and Trigonometry 6, 7, 8 G-SRT 6,7,8	Calculating Volumes of Compound Objects	15
M2 Geometric Modeling 2 G-MG Modeling with Geometry 1, 2, 3 G-MG 1,2,3	Modeling: Rolling Cups	10
	TOTAL:	144

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Algebra 2 Course Outline

Content Area	Formative Assessment Lessons	# of Days
A0 Introduction: Mathematical Investigations		8
A1 Exponential Functions N-RN The Real Number System 1, 2 A-SSE Seeing Structure in Expressions 3c, 4 F-IF Interpreting Functions 3, 7e, 8b F-BF Building Functions 2, 4a F-LE Linear, Quadratic, and Exponential Models 2, 4, 5 N-RN 1,2; A-SSE 3c,4;F-BF 2 → F-IF 3,7e,8b; F-LE 2,4 → F-BF 4a; F-LE 5	Rational and Irrational Numbers 2	25
A2 Trigonometric Functions F-IF Interpreting Functions 7e F-TF Trigonometric Functions 1, 2, 5, 8 F-IF 7e → F-TF 1,2,5,8	Ferris Wheel	15
A3 Functions F-IF Interpreting Functions 4, 6. 7c, 9 F-BF Building Functions 1a, 1b, 3 F-LE Linear, Quadratic, and Exponential Models 2, 5 F-IF 4,6; F-BF 1a,1b → F-IF 7c,9; F-LE 2 → F-BF 3; F-LE 5		26
A4 Rational and Polynomial Expressions N-CN The Complex Number System 1, 2, 7 A-SSE Seeing Structure in Expressions 2 A-APR Arithmetic with Polynomials and Rational Expressions 2, 3, 4, 6 A-CED Creating Equations 1 A-REI Reasoning with Equations and Inequalities 1, 2, 4b, 6, 7, 11 G-GPE Expressing Geometric Properties with Equations 2 A-SSE 2;A-APR 2,3; A-REI 1,2,11 → A-APR 6;A-CED 1; A-REI 4b → N-CN 1,2,7;A-APR 4; A-REI 6,7;G-GPE 2	Representing Polynomials Manipulating Radicals	15
P1 Probability N-Q Quantities 2 S-CP Conditional Probability and the Rules of Probability 1, 2, 3, 4, 5, 6, 7 N-Q 2 → S-CP 1,2,3,4,5,6,7	Modeling Conditional Probabilities 1: Lucky Dip Modeling Conditional Probabilities 2 Medical Testing	28
S2 Statistics S-IC Making Inferences and Justifying Conclusions 1, 2, 3, 4, 5, 6 S-IC 3,4,5,6 → S-IC 1,2		27
Total days:		138

Most of the Formative Assessment Lessons are available at http://map.mathshell.org/materials/lessons.php

Formative Assessment Lesson Overview

Algebra 1	Geometry	Algebra 2
Functions and Everyday Situations	Representing and Combining Transformations	Rational and Irrational Numbers 2
Modeling: Having Kittens	Analyzing Congruency Proofs	Ferris Wheel
Rational and Irrational Numbers 1	Applying Angle Theorems	Representing Polynomials
Comparing Investments	Evaluating Statements About Length and Area	Manipulating Radicals
Generalizing Patterns: Table Tiles	Inscribing and Circumscribing Right Triangles	Modeling Conditional Probabilities 1: Lucky Dip
Manipulating Polynomials	Geometry Problems: Circles and Triangles	Modeling Conditional Probabilities 2
Interpreting Algebraic Expressions	Proofs of the Pythagorean Theorem	Medical Testing
Sorting Equations and Identities	Solving Geometry Problems: Floodlights	
Building and Solving Equations 2	Finding Equations of Parallel and Perpendicular Lines	
Solving Linear Equations in Two Variables	Equations of Circles 1	
Optimization Problems: Boomerangs	Equations of Circles 2	
Defining Regions Using Inequalities	Sectors of Circles	
Forming Quadratics	Evaluating Statements About Enlargements (2D & 3D)	
Solving Quadratic Equations: Cutting Corners	2D Representations of 3D Objects	
Representing Data 1: Using Frequency Graphs	Calculating Volumes of Compound Objects	
Representing Data 2: Using Box Plots	Modeling: Rolling Cups	
Interpreting Statistics: A Case of Muddying the Waters		
Devising a Measure for Correlation		

Formative Assessment Lessons Aligned to Algebra 1

Lessons: A1 Modeling with Functions 15 Days

Functions and Everyday Situations

This lesson is about being able to model everyday situations using graphical and algebraic representations of real-valued functions. Students need to translate between the situations described, the graphs, and the algebraic formulas for the functions that serve as models.

In the first four and the last four situations presented to them, students need to produce appropriate graphs that show the relationship between the two quantities being described. They need to consider whether each graph should be discrete or continuous based on the context. Then they are given formulas for the functions that model the situations, and they need to match each to its corresponding situation and graph. Finally, they are asked several questions requiring them to interpret the graphs in the context of the situation described.

There are also twelve "everyday situation" cards that students need to match with twelve graph cards, and later on each pair of cards needs to be matched to an equation card.

There are several functions used to model the many situations of this lesson: linear (through the origin and otherwise), quadratic, exponential, rational, radical, trigonometric. Some contexts are discrete, some are continuous, and some are step functions with one variable discrete and the other continuous. Students need to have a working knowledge of how to handle a variety of such functions, their graphs, their domains, and their properties. They also need to know how these properties make them good models for certain situations. Finally, they need to know or figure out a few facts about the everyday situations presented to them, for instance that the radius of a sphere will grow as the cube root of its volume, or the laws of physics governing free falling objects thrown with a given initial velocity, or that in some situations the product of the quantities described must be a constant.

Modeling: Having Kittens

This lesson is a very rich modeling task. A claim is made about the possible number of descendants of a (female) cat in an 18-month period. Students need to decide whether or not the claim is reasonable. They are given some information, like the average number of litters a female has in one year, the average time of pregnancy, the average size of a litter, and the age range over which a female cat can have kittens.

Students need to decide what pieces of information to use, and they need to explicitly state the assumptions that they will make in order to model the situation and figure out an estimate for the number of descendants. For example, they may make the assumption that all kittens in the model are female, in order to avoid complications about descendants of male cats, or they may assume otherwise and deal with descendants of males. Students need to keep track of the litters of every new generation, and they need to realize the exponential nature of the growth. They need to make posters with complete explanations and diagrams, and they need to explain it to others and to critique others' posters, noting clarity, completeness, strengths, and weaknesses of the different solutions. They also need to critique three given sample solutions, each with different flaws (for example, counting only the kittens born from the original cat). If they choose to model the problem algebraically, they will need some familiarity with exponential functions.

Rational and Irrational Numbers 1

In this lesson, students need to classify several real numbers given in different forms (like fractions, decimals, repeating decimals, square roots, and arithmetic combinations of some of these), as rational or irrational. In some cases the notation or description of the number does not provide enough information to make the classification, and students need to discover these cases.

In order to succeed, they need to know the definition of rational (and irrational) number. They also need to have an agreement as to what a "fraction" is, especially that a "fraction," as used here, is not equivalent to a rational number. Finally, they need to know for a fact that the square root of an integer that is not a perfect square is an irrational number. They are not required to prove this. They should know that the rational numbers are closed under the arithmetic operations, and be able to use this fact (via an easy contradiction argument) to establish results such as that the sum of a rational and an irrational number must be irrational. They need to be able to manipulate terminating and repeating decimals and write them as a quotient of two integers. But they are not required to formally explain the meaning of the repeating decimals' notation or the arithmetic manipulations performed on them. That is to say, they are not asked to deal with infinite series. Finally, they need to simplify products or sums in order to establish the rationality or irrationality of some numbers.

Students are given several numbers and categories of decimals to place them under. These categories are: terminating decimals, repeating non-terminating decimals, and non-repeating non-terminating decimals. They need to realize that terminating and repeating decimals are always rational, while non-terminating and non-repeating decimals are always irrational.

Lessons: A2 Linear Functions
16 Days

Comparing Investments

This lesson is about investments with simple and compound interest rates. The main mathematical tasks are to compare the linear versus exponential models that arise from these investments, and to be able to represent and describe them in a variety of ways, including verbal descriptions, algebraic equations, tables of values, and graphs.

Students are asked to compare a simple interest situation (linear) with a compound interest situation (exponential). They need to find equations for each, and match the situations to the appropriate graph among a few that are provided. They also need to compare investments of each type over shorter and longer time intervals. By identifying which among three simple interest situations is the "odd one out," students investigate the features of each investment plan and how they affect the investment over time, as well as how they turn up in the equations. They need to do the same with three compound investment situations. After this, they are required to match verbal descriptions with equations, tables and graphs. Finally, they are asked to match these with additional descriptions comparing the investments in terms of the time they take to double the initial amount of money or how much money they yield over different time intervals. In order to engage in this lesson, students need to be familiar with linear and exponential functions, their properties, and their graphs.

Generalizing Patterns: Table Tiles

Through the context of tiling square tabletops of different sizes, students need to discover, generalize, and prove the correctness of three sequential patterns. One of the patterns yields a constant sequence, one yields a linear (arithmetic) sequence, and the third one yields a pattern described explicitly by a quadratic function.

Students may obtain recursive formulas to discover the way in which the first and second consecutive differences behave. If they do, they need to justify the recursive formulas, and, more importantly, they need to use them in order to find explicit formulas that generalize their results for tables of arbitrary size. Students need to understand the difference between a conjecture and a formal argument that proves the conjecture.

They need to find and organize data, make predictions or conjectures, and prove them. Students are also asked to criticize sample solutions for correctness and completeness of the arguments. This lesson demands some familiarity with linear and quadratic equations, and with the behavior of first and second consecutive differences in tables of values coming from linear or quadratic functions evaluated at consecutive integers.

Lessons: A3 Linear Equations & Inequalities in One Variable 16 Days

Manipulating Polynomials

In this lesson, students are given the first few terms of several sequences of rectangular arrays with patterns of black and white dots. They need to recognize and extend the patterns, and find algebraic expressions for the number of black dots, the number of white dots, and the total number of dots in the nth term of each sequence. They also need to verify that the sum of the number of black and white dots equals the total number of dots in the nth array. Students are also given several cards with diagrams of arrays and several cards with algebraic expressions for the number of dots (black, white, total), and they need to match them and explain their reasoning. Finally, they need to create a few patterns of their own and their corresponding expressions.

As the patterns involved are all linear or quadratic, students need to be able to recognize such patterns and represent them algebraically as expressions, as well as manipulate these expressions formally to verify that the sums are correct. They must also be able to visualize these expressions as geometric patterns in order to go back and forth between the diagrams and the algebraic expressions.

Interpreting Algebraic Expressions

In this lesson, students need to write a one-variable expression from a description of the expression given in words. They are then shown four sample equations, and asked to determine if the expressions on both sides are actually equivalent, and to replace one if they are not. They also need to match four sets of cards for linear and quadratic expressions in one variable: symbolic expressions, word descriptions, tables of values, and area models. Students need to be familiar with the distributive property and order of operations, and to use them to recognize equivalent expressions.

Sorting Equations and Identities

In this lesson, students are asked to both write and classify equations based on the number of solutions in the solution set. Students will write equations that have no solutions, some solutions (one or two), and infinite solutions. Students will also classify equations into sets that are never true, sometimes true, or always true. Except for the case where the solution set is an infinite but proper subset of numbers, these classification systems are two ways of saying the same thing. Students will develop the notion of an "identity" as an equation that is always true.

The families of equations that students will work with are primarily linear and quadratic, with a few variations such as an equation in two variables and a proportion. Some linear and quadratic equations are chosen to highlight common mistakes students make when applying the distributive property, including multiplying binomials; however, these skills are not the intended focus of the lesson. The logic that students apply to categorize the equations is more central.

A variety of methods may be used to classify equations, so the approaches a student uses may vary by equation. For example, a student may choose to use substitution to reason that an equation is "sometimes true" by showing it has both solutions and non-solutions. Since classifying is the mathematical focus of the lesson, any correct method that occurs to the student is acceptable.

Note: Students who have studied imaginary numbers can be expected to use that knowledge here as the answer key is easily adjusted.

Building and Solving Equations 2

This is a lesson about solving linear equations in one variable. The variable can appear on one or both sides of the equation. Students are required to record and explain each step, and once they obtain solutions, they need to check them by reversing the steps of the solution to obtain the original equation, as well as substituting the solution value into the equation.

In order to succeed, students need to know and apply the order of operations and the distributive law. They also need to understand and use (in order to simplify the equations) the fact that adding or subtracting the same number to both sides of an equation results in a logically equivalent equation, as does multiplying/dividing both sides by the same nonzero number.

Lessons: A4 Linear Equations & Inequalities in Two Variables
16 Days

Solving Linear Equations in Two Variables

The goal of this lesson is for students to gain an understanding of how linear equations in two variables can be used to model real-life situations. In the lessons, students will interpret the meanings of linear equations in two variables in different contexts. They must also critically analyze incorrect interpretations of linear equations and explain why the interpretations are wrong. They will use and analyze several methods in the course of the lesson: elimination, substitution, guess and check and graphing the equations to find the intersection.

Optimization Problems: Boomerangs

This is a lesson about optimization. Students need to maximize a profit, which can be represented by a linear function of two variables. The constraints on the variables yield a feasible region bounded by the axes and two linear equations, resulting in a convex quadrilateral in the first quadrant. Students may graph the feasible region, which involves solving a system of linear inequalities in two variables. Once they have the feasible region, they may test the lattice points contained in it to find the maximum profit (the context makes the problem discrete, so only lattice points can be allowed as solutions). They may know or discover that in fact the maximum must be attained at a vertex of the feasible region, if the vertices are all lattice points, which is the case here. Students may instead organize data on a table to try to find the maximum profit.

Students are also given several sample responses, each with different strengths and weaknesses, and asked to analyze and critique them. To succeed in this lesson, students must apply what they know about linear equations to model a realistic situation, representing the constraints and variables mathematically.

Defining Regions using Inequalities

This lesson is about systems of linear inequalities in two variables. Students need to substitute values for *x* and *y* in the inequalities to find out if certain points belong to the solution set, i.e., the region (or lattice points in the region, since the solutions are lattice points by convention) where all the inequalities are simultaneously satisfied. They need to be able to graph linear equations and inequalities, and graph the region that forms the solution set of a system of inequalities. They also need to determine the usefulness of a certain clue, given a number of previous clues, and understand that some clues may be superfluous, while the most useful clues are the ones that restrict the solution set the most. Finally, students use a prescribed list of inequalities to construct a problem with a unique lattice point as its solution.

Lessons: A5 Quadratic Functions
20 Days

Forming Quadratics

The goal of this lesson is for students to understand the different types of equations that describe quadratic functions, and the information that the different equations readily supply. In the standard form, the constant term gives the *y*-intercept, and the sign of the leading coefficient determines whether the function has a minimum (the graph is a parabola whose vertex is its lowest point) or a maximum (the graph is a parabola whose vertex is its highest point). In the factored form (factoring over the reals), the factors give the (real) roots, if there are any. These are the *x*-intercepts of the graph. A squared linear factor means that there is a double root, and the vertex of the parabola lies on the *x*-axis. The vertex form gives the coordinates of the vertex, and can also be easily used to decide if the function has real roots or not, because it shows whether the function has a minimum or a maximum, and also because this form contains a completed square.

Students need to be able to graph quadratic functions and to evaluate them at a given number. They need to be able to transform an equation for a quadratic function from one type to another. For this, they must be able to multiply linear factors, factor quadratic expressions, and complete the square. They need to match graphs to equations and vice-versa.

Lessons: A6 Quadratic Equations 26 Days

Solving Quadratic Equations: *Cutting Corners*

In this lesson, students need to figure out the distance that the back wheel of a bus cuts into a bike lane as it turns with its front wheels going over the outside edge of the bike lane. The information given is the distance between the front and back wheels, and the radius of the outside of the bike lane.

Students need to make a drawing modeling the situation, which leads to a right triangle. They must use the Pythagorean theorem in order to obtain a quadratic equation involving the given information and the distance they need to find. Finally, they must solve this quadratic equation. They can complete the square or use the quadratic formula, and they can also make a linear change of variables leading to an easier quadratic equation. Once they solve this equation, they must interpret the algebraic solutions in the given context in order to keep the correct value and discard the other, which is larger than the outside bike lane radius and hence not what they are looking for.

Next, students need to find how far the front wheel of the bus must be from the outside edge of the bike lane in order for the back wheel not to cut into the lane. This requires a similar procedure, involving the same ideas as above, but the variables change slightly. Students are also asked to analyze, compare, and evaluate four solution attempts provided to them as sample student work.

Lessons: S1 Statistics 30 Days

Representing Data 1: Using Frequency Graphs

This lesson and *Representing Data 2: Using Box Plots* are very similar, and they share some materials for students to work on. The goal is for students to interpret and analyze data about the frequency of an event using frequency graphs and verbal descriptions, and connecting the two representations.

Although the data are finite and hence discrete (about a group of students and their monthly cell phone spending or their scores on a test), students are presented with continuous frequency graphs of the situation, and they need to understand how the continuous graph approximates and relates to the real, discrete data. For example, they need to approximate the area under the graph to make deductions about the sample size (number of students) and about the median value. For making these approximations, they may need to triangulate the region under the curve or make a grid of thin rectangles. They are then asked to determine the mode and range of the distribution from the graph. They also need to make inferences about the sampled students or about the test given the shape and range of the frequency graph. Finally, students are asked to match a set of graphs to a set of verbal descriptions in the context of a test.

In order to succeed in this lesson, students must have an understanding of mean, median, mode and range, and be able to estimate them from frequency distributions.

Representing Data 2: Using Box Plots

This lesson goes hand in hand with *Representing Data 1: Using Frequency Graphs*. It is also about data analysis and interpretation of data. But whereas in *Representing Data 1: Using Frequency Graphs* students are asked to match frequency graphs to verbal descriptions, in this lesson they have to match the same frequency graphs to their corresponding box plots.

Students need to understand what information is provided, as well as what information is *not* provided (for example mean, mode, sample size, distribution within each quartile) by a box plot. They also need to understand that there is a unique box plot that corresponds to any given frequency graph, but there are many frequency graphs that would produce any given box plot. Students are asked to draw a box plot corresponding to a given frequency graph, and to draw more than one frequency graph with varying features that all correspond to the same given box plot. They are also asked to interpret the box plots and graphs in the context of the data (a group of students who took a test). Again, students need to be able to estimate and to compare areas under the frequency graph in order to approximate the quartiles for the corresponding box plot.

Interpreting Statistics: A Case of Muddying the Waters

This lesson is about interpreting data. Students are given, and asked to act out, complex scenarios resembling real-world situations, where there is a dispute to be resolved in court.

Data are presented to them in several forms, including diagrams, graphs, surveys, tables, pie charts, and scatter plots. Students need to evaluate the claims made on both sides of the arguments, and they need to think critically about the inferences or conclusions drawn from the data by the different parties. In order to do so, they must attend to several issues that can result in misleading or manipulative data, like bias of samples, bias of questions, and scaling of graphs. They need to consider the importance of the size of samples and surveys and whether or not they can be biased because of the way they were collected. Students also need to evaluate claims about causal relationships based on the data. They need to consider plausible alternative causes not shown by the data. Overall, the lesson demands analytical and critical thinking, as well as communication and debating skills.

Devising a Measure for Correlation

This is a lesson about mathematical modeling in the context of statistical correlation between two variables. Students are asked to describe three different scatter graphs showing several data points. One graph shows a strong correlation, another shows a weak correlation, and the third shows no correlation. They are also presented with a scatter graph with fewer points, and asked to provide additional data points that would increase or decrease the correlation between the variables.

Students are asked to carefully analyze four different models for obtaining a correlation coefficient: one in the first assigned task, and three other methods provided as sample student work, after they have attempted to come up with models of their own. They need to evaluate and compare all the models (those provided as well as their own and each other's). The methods provided include considering the area enclosed by the convex hull of the data points, looking at the vertical distance between data points and an estimated line of best fit, splitting the graph into four equal rectangular regions and counting the number of data points in them (two regions correspond to strong correlation and the other two to weak correlation), and comparing the *x*- and *y*- ranking (order) of the data points.

Students need to consider, for each model, what range of values the method can yield, examples of data that may not be well accommodated by the model, sensitivity to outliers, and the result of changing scales on the axes. Students are not required to know a formula for the correlation coefficient, but they need to have a common-sense understanding of its meaning and they need to be able to describe the correlation by seeing the scatter graphs.

Formative Assessment Lessons Aligned to Geometry

Lessons: G1 Basic Definitions and Rigid Motion 20 Days

Representing and Combining Transformations

This lesson is about transformations that preserve distance in a coordinate plane – translations, rotations, and reflections. These transformations are also called rigid motions or congruence transformations because they preserve the size and shape of geometric figures, although they may change the orientation of the figure in the plane.

In this lesson, students apply transformations to geometric images and describe transformations that carry one image onto another. In the collaborative activity, the students consider how to link two images with a transformation and explain their reasoning. Transformations can be performed by using the right "motion" along a particular path. In a translation, all points move vertically or horizontally for the given number of units. In a rotation, all points move in a circle about a fixed point (the circle's center). In a reflection, points "flip" over the line of reflection along a line that's perpendicular to it. In a rotation and reflection, distances to the reference point or line must stay fixed.

When students begin linking images with transformations, they are (whether they realize it or not) working with transformations as functions and treating the images as inputs and outputs. In searching for all possible links that can be made, students re-use outputs as inputs and create sequences (compositions) of transformations. Students may be asked to look for sequences of two or more transformations that can be achieved with a single transformation. As they do this, students are actually checking for compositions of functions that are equivalent to other functions (for a given input).

The description of transformations using an algebraic rule arises out of performing them using the appropriate motion. In this lesson, using the algebraic rules that apply to the coordinates to *perform* the transformations is not a central focus.

Analyzing Congruency Proofs

In this lesson, students need to decide if two triangles sharing a given number of side lengths and a given number of angle measures must be congruent. First they are asked about three specific examples, and then they are provided with nine cards, each of them stating that two triangles share *n* side lengths and *m* angle measures, where *n* is 1, 2 or 3 and *m* is 0, 1 or 2. Hence they have nine cards (all combinations above are given). For each card, they need to decide if the triangles must be congruent or not. In the former case, they must prove it, and in the latter case they must find a counter example, and if possible, a further explanation of why the conditions are not the same as ASA or SAS conditions that do guarantee congruence. For this point, students must understand that the order of the corresponding sides and angles in ASA and SAS matters.

In order for students to access this lesson, they need to know some basic logic, like the fact that one counterexample is enough to establish the falsehood of a (universal) statement, while using inductive reasoning based on any number of confirming examples is not enough to prove such a statement. They also need to know the basic triangle congruence theorems, (SSS, SAS and ASA) and they need to know that SSA or AAS is not sufficient for congruence (although it IS sufficient in SSA if the angle is opposite the longer side, a result little known in school). Students also need to know similarity conditions, most crucially AA, and the fact that since all triangles share the same angle sum of 180 degrees, sharing two angle measures is equivalent to sharing all three of them.

Lessons: G2 Geometric Relationships and Properties 15 Days

Applying Angle Theorems

In this lesson, students are given problems that require them to find interior and exterior angles of polygons. They are given several sample solutions to the problem which they are asked to analyze and criticize. In analyzing these sample solutions, students need to recognize whether false assumptions are being made, or if there are computational errors.

Students are given a sheet with relevant terminology and facts about angles formed by lines, angles formed by a transversal to parallel lines, the interior angle sum of a triangle, and the interior and exterior angle sums of polygons. In order to criticize some solutions, or to come up with their own, students need to be familiar with triangle congruence theorems and basic facts about parallelograms. They also may need to add auxiliary lines in a given figure.

Evaluating Statements about Length and Area

This lesson is about the perimeter and area of different kinds of figures: quadrilaterals, triangles, and circles. Students need to determine whether certain statements are always, sometimes, or never true, and prove their assertions. When the statements are always or never true, the proof entails providing an argument that applies to any figure of the kind being considered. When the statement is sometimes true, students need to provide examples, counterexamples, and, more importantly, the extent to which the statement is true. That is, they need to restrict the set of figures being considered so that the statement will always be true for those figures, and false for others.

In order to complete this lesson, students need to be familiar with the formulas for the perimeter and area of triangles, trapezoids, parallelograms and circles. They also need some experience proving and disproving geometric statements, as well as some experience with logical reasoning and proofs. They also need to know the congruence theorems for triangles, basic facts about the properties of parallelograms, and the fundamental theorem of similarity.

Inscribing and Circumscribing Right Triangles

This is a lesson about finding the radii of both the circumscribed and the inscribed circles of a right triangle. Students are asked to find them in a specific example, a 5-12-13 triangle that is not drawn to scale, and later they need to generalize their results to arbitrary right triangles. Besides the usual small group collaboration, they are given a few questions to reflect on how they did individual and collaborative work.

There are a few different ways to solve the problem. In order to do that, and to evaluate the sample student work that is given to students, they need to be familiar with the circle and triangle theorems of Geometry. Four circle theorems are provided, two of which are most relevant here: the one about the measure of an inscribed angle being half the measure of the corresponding central angle, and the one stating that tangent segments from a common point are congruent. Some of the triangle theorems and other supporting Geometry facts that students should know and be able to apply to this problem are the definition of distance between a point and a line; that a point on an angle bisector is equidistant from the sides of the angle; that the angle bisectors of a triangle are concurrent; that every triangle has a unique circumscribed circle (its center is the point of concurrency of the perpendicular bisectors of the sides); that every triangle has a unique inscribed circle (its center is the point of concurrency of the angle bisectors of its angles); and the formula for the area of a triangle given a side and the altitude from the opposite vertex.

Lessons: G3 Similarity
20 Days

Geometry Problems: Circles and Triangles

This Geometry lesson asks for the ratio of the areas of two equilateral triangles, one circumscribed and the other inscribed in the same circle. Then it asks for the ratio of the areas of two circles, one circumscribed and one inscribed in the same equilateral triangle.

There are many different ways to solve this problem. Students may use or derive facts about equilateral triangles, like the fact that their medians cut each other in segments of ratio 2 to 1. Depending on their approach, they may need some basic right-triangle trigonometry, or else knowledge of the ratio of side lengths of 30-60-90 triangles. They may need to know how to derive the area of an equilateral triangle given its side length, and they may need to know the formula for the area of a circle. They may also need to know about similarity and the effect on areas upon scaling by a given factor. They may use rotations and triangle congruence theorems. Finally, students are asked to examine carefully and criticize several sample solutions for correctness, clarity, and completeness.

Proofs of the Pythagorean Theorem

In this lesson, students need to use diagrams provided to them in order to prove the Pythagorean theorem in different ways. They also need to analyze, evaluate and correct different proofs that may have mistakes, given as sample student work. Together, they provide several different ways to prove the theorem.

Students need to be able to translate back and forth between the visual information of the geometric diagrams and the algebraic relationships that they reveal, through the consideration of lengths, areas, congruence and similarity. They also need to be able to explain why a geometric figure really is what it looks like in a diagram (for example, a square or a trapezoid) by considering lengths and angles. They have to be able to label lengths of segments in a useful way, and to visualize how one diagram can be changed into another by moving around congruent pieces, in order to represent certain quantities in two different ways.

Lessons: M1 Geometric Modeling 1
7 Days

Solving Geometry Problems: *Floodlights*

In this lesson, students are asked to solve a geometric problem given as a word problem. A person stands between two floodlights that are at the same height. Students need to find the length of the shadows and hence the total length of the two shadows. Then the person starts moving toward one of the floodlights, and students are asked to explain what happens to this total length.

First, students need to be able to model this situation with an appropriate geometric picture. They also need to recognize that the key to the problem (that the length of the combined shadows remains constant) is similar triangles, and once they find useful similar triangles, they need to use the facts about ratios of corresponding sides being equal. In order to establish the similarity of the triangles, students need to know that opposite sides of a rectangle are parallel, that alternate interior angles cut by a transversal to two parallel lines are equal, and that triangles sharing two angle measures are similar.

Lessons: G4 Coordinate Geometry
15 Days

Finding Equations of Parallel and Perpendicular Lines

In this lesson, students work with sets of linear equations written in different forms to sort out pairs of parallel and perpendicular lines from other pairs of intersecting lines. Pairs of lines that are coincident are not considered here.

In the assessment, students sort out the pairs of parallel and perpendicular lines in the process of looking for four equations whose graphs intersect to form a rectangle (as it turns out, two of the vertices occur on the x- and y-axes). In the lesson, students sort pairs of equations into categories based on whether their lines are parallel, perpendicular, or intersect at a particular point (y-intercept, x-intercept, or another point).

In their search, students will use the equations of the lines as a starting point for figuring out how they relate to each other. Students who rely solely on the way the equation is written to make decisions may or may not be successful. After all, there is no guarantee that pairs of equations *written in a particular form* will have the relationship the student is looking for, unless the form is one that reveals key information about the lines. Slope-intercept form is especially useful for spotting parallel and perpendicular lines, while other forms may prove more useful for deciding if and how lines intersect.

Therefore, the mathematical focus of this lesson is not on the *skill* of “converting to slope-intercept form to classify pairs of linear equations”, but rather on the recognition of the usefulness of forms of equations that reveal information. Students may use all connections between representations of lines (algebraic, graphical, and numerical) that they find helpful for determining the relationships between the linear equations. For instance, it is not necessary to change the form of an equation to decide if the line contains a desired intersection point.

G5 Circles and Conics
15 Days

Equations of Circles 1

In this lesson, students need to find an equation for a circle given some information about its graph, such as the coordinates of the center and the radius. They also need to be able to graph a circle given its equation, and find the possible coordinates of a point on the circle given the other coordinate. The equations are all given (with only minor changes like the order of the terms) in the form where the squares have been completed, so that the center and radius are immediately apparent from the equation. Students are given several such equations, and they have to correctly place each one in a matrix of different center locations and radii. In order to understand the lesson and find points on circles, as well as the equations, students need to know the Pythagorean theorem and the distance formula, be familiar with the Cartesian (coordinate) plane, and be able to manipulate square roots.

Equations of Circles 2

This lesson is also about circles and is similar to *Equations of Circles 1*. Roughly the same skills, knowledge and ability are necessary. Again the equations given or asked for are already in completed squares. In this lesson, students need to find the *x*- and *y*- intercepts (if any) of circles. They are then given several equations and graphs, they need to place each in a chart that is a matrix with the number of *x*- and *y*- intercepts that a circle may have (0, 1, or 2 of each). They also need to find equations for circles having certain properties and a given number of *x*- and *y*- intercepts, and they have to explain why the equations they come up with produce circles with the required properties. Another task of the lesson involves considering three circles and finding different criteria under which any one of them could be considered the odd one out.

Sectors of Circles

In this lesson, students study sectors of circles, paying attention to their angle, radius, arc length, and area. Students are led through a discussion of radians, arc length, and area of sectors.

Students need to find the formulas for arc length and area by understanding how the angle of the sector determines the fraction of a circle that it makes up, and then use proportional reasoning together with the formulas for the circumference and area of a circle. Given three concentric circles with specified radii (in the ratio 1:2:4), they need to find different sectors on any of the circles that have the same area or arc length. For this, they need to understand what happens to lengths and areas when scaling figures. They also need to understand, using the formulas they obtained, how the arc length and area of a sector change with its angle and with its radius.

Students are then asked to sort domino cards into a loop, where half of each card is a sector (whose arc length, radius, perimeter and area they need to compute given one of these quantities) and the other half is a set of instructions for changing or maintaining some of these quantities. They also need to match cards that describe changes in arc length or area with cards that describe changes in angle and radius to produce the changes given in the first set.

Lessons: G6 Geometric Measurements and Dimensions
15 Days

Evaluating Statements about Enlargements (2D & 3D)

This lesson is about the change in perimeter, area, and volume resulting from scaling two-dimensional and three-dimensional figures. Several statements are made about the change in perimeter, area, or volume of figures resulting from applying a scale factor to a linear dimension of the figure. The 2-D figures are rectangles and circles, and the 3-D figures are rectangular prisms, spheres, right cylinders, and cones. The required formulas for the volume and lateral area of solids are given to the students, as are the formulas for the circumference and area of a circle.

Students need to decide, for each given statement, if it is true or false. They need to explain their reasoning to each other, and if a statement is false, they are asked to change it so as to make it true.

The purpose of the lesson is for students to understand that if two similar figures are related by a linear scalar factor *k*, then corresponding linear measurements (like perimeter) will be related by the same factor *k*, while corresponding area measurements (like surface area or lateral area) will be related by a factor k^2 and corresponding volumes will be related by a factor k^3. But students are also asked to consider what happens to perimeters, areas and volumes if some dimensions are scaled, but not all (breaking similarity).

2D Representations of 3D Objects

This lesson is about the visualization of planar cross sections of three-dimensional objects. The 3D objects themselves are not provided physically, but as 2D drawings that represent them. Students therefore need to visualize the 3D objects from the drawings, and then also visualize planar cross sections of the objects. Some of the 3D objects represented by drawings are spheres, hemispheres, cones, tetrahedrons, octahedrons, cubes, rectangular and triangular prisms, rectangular pyramids, and cylinders.

Students need to consider what the surface of the water would look like as a 3D object, placed in a certain rigid orientation, is being drained. They need to draw the cross sections at several distinct moments. The orientation of the object matters (in most cases—not for a sphere), and students need to realize this.

Later, students are provided with cards that they need to match. Six of the cards contain drawings of two identical 3D objects, one on top of the other, positioned with different orientations, and connected with a thin vertical pipe. In each case, the top object is initially filled with water and starts draining into a lower, identical object. Twelve cards contain drawings with possible sequences of cross sectional shapes for either the top or the bottom object in the six cards. Students need to map each of the twelve sequences to its corresponding object, as it is being drained (top) or filled (bottom). Some of the cross section cards are missing drawings in the sequence, and students need to supply the missing cross sections. They need to explain their drawings and their matching.

This lesson requires a good deal of visualization, communication, and justification skills, as well as drawing ability. Students should also know the basic properties of common 3D and 2D geometric figures and be able to use them in their argumentation. Finally, they need to understand what is meant by a planar cross section, i.e., the intersection of the 3D object with a plane (in this lesson, a horizontal plane), and they must know that two non-parallel planes intersect along a line. A useful fact for the lesson is that if a plane intersects two parallel planes, the resulting lines are parallel. Students can apply this fact to deduce properties about the possible cross sections of prisms (for example, cubes).

Lessons: G7 Trigonometric Ratios
15 Days

Calculating Volumes of Compound Objects

In this lesson, students need to find the volume of different-shaped glasses by decomposing them, if necessary, into simpler shapes whose disjoint union makes up the whole glass. These shapes are cylinders, hemispheres, and cones. Students are given the necessary volume formulas. Students are also asked to find the height of the water in some of these glasses when the glasses are half full. This involves more demanding reasoning and computation. In the case of a cone, students need to make use of similar right triangles and proportional reasoning to obtain a relationship between the radius of the surface of the water and the height of the water in the glass.

Students are also asked to criticize and correct sample solutions. Finally, one multiple-choice question asks students to think about dimensional analysis in order to choose a correct volume formula, and to consider whether it makes sense for a given height or a given diameter to be squared in that formula.

Lessons: M2 Geometric Modeling 2
10 Days

Modeling: *Rolling Cups*

This is a lesson in which students need to analyze videos and data, make their own hands-on and computer-based experiments, and make a mathematical model, in order to figure out the roll radius of any cup (in the shape of a frustum) given its wide diameter, narrow diameter, and slant height. Students need to think about the relationship between the roll radius and the three given measures. They need to discover the idea of fixing two of these three measures at a time and varying the third, and explain what happens in these cases. They also need to reason that if the wide and narrow diameter are very close to each other, the roll radius is large, and in the limiting case when the diameters are equal, there is an “infinite” roll radius and the cup rolls in a straight line rather than a circle.

A complete solution requires drawing the vertical cross section of a cup and extending this drawing to a cone, in order to use similar triangles and the resulting proportional relationships in the figure. The roll radius, which is the slant height of this cone, can then be obtained in terms of the three given measures of the cup. Once students have such an expression for the roll radius, they can study algebraically how it varies with each of the measures, and they need to confirm that these results agree with their experimental ones.

Formative Assessment Lessons Aligned to Algebra 2

Lessons: A1 Exponential Functions
25 Days

Rational and Irrational Numbers 2

This lesson is also about rational and irrational numbers. Students are asked for examples of rational and irrational numbers, and for a definition in their own terms. Students need to explore what happens when adding, subtracting, multiplying or dividing two rational numbers, two irrational numbers, or one of each. They need to prove or disprove statements about the resulting numbers by using the fact that the rational numbers are closed under the arithmetic operations, using contradiction arguments, and finding counterexamples. They also need to consider lengths and areas of geometric objects like circles, right triangles, and rectangles, and find examples of when one or more of these measures can be rational or irrational.

Students are also given cards with various statements related to rational and irrational numbers, and asked to sort the statements into three categories: always true, sometimes true and never true.

Lessons: A2 Trigonometric Functions
15 Days

Ferris Wheel

In this lesson, students model how the height above the ground of a Ferris wheel rider changes over time as the wheel turns at a constant rate. Students need to use a trigonometric function to model this behavior, in this case, a cosine function. They are given the relevant information about the Ferris wheel: the height of the axel, the diameter of the wheel, the starting point of the rider, and the period of the rotation. Students are then asked to convert these values into the basic parameters of a sinusoid: amplitude, period, phase and offset. They use this information to graph the position of the rider over time.

The goal of this lesson is to improve students' understanding of the relationships between the physical properties of rotating objects and the mathematical parameters of trigonometric functions.

Students are then paired and asked to match expressions of several cosine functions of varying amplitudes, offsets, and periods with corresponding graphs (if any graph is missing, they need to draw it). Finally, they are asked to match these with corresponding descriptions of several Ferris wheels of varying diameters, axle heights, and periods of revolution. In order to accomplish these tasks, students should already be familiar with the sine and cosine functions and with how the parameters of amplitude, frequency, and shifts affect their graphs. They also need to know that one can model the height of a point on a circle turning at a constant rate as a cosine (or sine) function. Ideally, they should know that the *x*- and *y*- coordinates of a point on the unit circle are the cosine and sine (respectively) of the angle between a radial line through that point and the positive *x*-axis.

Lessons: A4 Rational and Polynomial Expressions 15 Days

Representing Polynomials

Apart from a short consideration of linear and quadratic functions in the teacher guide and projector materials, this lesson deals exclusively with cubic functions. They are used to illustrate the relationships between zeros, factors, equations, and graphs, and between shifts (vertical and horizontal) and reflections (across the *x*-axis and the *y*-axis) and the corresponding changes in the algebraic expressions of the (cubic) functions. Students need to understand and apply these ideas in order to answer several questions, match equations to graphs (sometimes more than one equation can correspond to the same graph), graph equations and change expressions in order to produce shifts or reflections. Students will also decide if statements dealing with zeros, factors, values at other values of *x*, and transformations (shifts and reflections) are true or false.

Students need to know that a constant a is a zero of a polynomial if and only if $(x - a)$ is a factor of the equation. They need to understand how double or triple roots affect the graph. They need to understand that changing x to $(x - k)$ in $y = f(x)$ results in a horizontal shift, changing $f(x)$ to $f(x) + k$ results in a vertical shift, changing $f(x)$ to $-f(x)$ reflects the graph over the *x*-axis, and changing x to $-x$ reflects the graph over the *y*-axis.

Manipulating Radicals

In this lesson, students are given several equations in one variable involving square roots (on two occasions, a second variable is used). In each case, they need to solve the equation and find whether the solution set is empty (no solutions), a proper subset of the real numbers (in the examples given there is only one or sometimes only two solutions), or all real numbers (the equation is an identity). These cases are referred to as saying that the statements given by the equations are never true, sometimes true, or always true.

In order to accomplish their task, students need to know about the laws of exponents, especially squares and square roots (exponent ½). They also need to know how to square binomials, how squaring and taking square roots are related, and how to isolate square roots and square expressions in an equation in order to solve it. There is also an optional extension, consisting of the last two equations provided, that explores the imaginary number i, the square root of -1. But issues of the domain of a function are not explored in any significant depth.

Modeling Conditional Probabilities 1: *Lucky Dip*

This is a lesson about conditional probability focusing on a single problem, a game in which two balls are randomly selected without replacement from a bag containing 3 black and 3 white balls. Students need to find the probability of selecting two balls of the same color. Students need to solve the problem, discuss each other's solutions, and analyze and critique three given sample solutions with different misconceptions. In order to access the problem, they need to know and build on the basic computation of probabilities given finite sample spaces where the outcomes are equally likely. They need to be able to represent the sample space and all its outcomes, or make probability trees, and they need to interpret such models in the sample solutions provided and find where they are flawed. Also, different models emerge depending on whether the student thinks of taking two balls at once, or of taking one ball after the other.

Modeling Conditional Probabilities 2

In this lesson, students are asked to consider a series of extensions of the problem of *Modeling Conditional Probabilities 1: Lucky Dip*. First, given a total of 9 balls, they need to find the two (symmetrical) solutions where the game would be fair, i.e., the number of white and black balls so that the probability of choosing two of the same color equals the probability of choosing two of opposite color. Then they are given different initial numbers of black balls, and they are asked to find the number of white balls they need to add in order to make the game fair. As it turns out, sometimes there is a unique solution, sometimes there are two, and sometimes there are no solutions. Finally, they are asked to find general relationships among the number of black balls, white balls, and their sum if the game is fair.

Students need to represent the sample space, count the outcomes with same color and different color, and equate these numbers in order to make the game fair. In order to solve the more general or demanding extensions, they will need to introduce literal symbols and use algebra, eventually leading them to solve quadratic equations (which can be factored easily when there are solutions). For the simpler tasks, they may also be able to obtain solutions by guess-and-check methods. Whether or not they can solve the problem in its most general, abstract form, they should notice some patterns and relationships provided by the numerical solutions to various examples. These patterns involve perfect squares and triangular numbers, which should be familiar to them.

Medical Testing

This is a lesson about conditional probability in the context of false positives in medical testing. Students are given the probability that a person without a certain disease will test positive (and they are told that a person with the disease always tests positive). Then they are asked to compare two samples of 1000 people, each with a different percent of people having the disease. For each sample, they need to compute the number of people having and not having the disease, and the number of people testing positive and negative. Finally, for each sample, they need to compute the probability of getting a false positive. For this, they need to realize that they need to compute the ratio of the number of people not having the disease and testing positive to the number of people testing positive (this last number, of course, includes all the people who actually have the disease).

Students need to make sense of the “false positive paradox,” namely that the probability of getting an incorrect positive test depends not only on the accuracy of the test but also on the rarity of the disease in the population. The rarer the disease is, the higher the probability that a positive test is wrong.

Students can solve the problem by modeling it in a variety of ways, and they are also asked to analyze given sample student work with varying approaches, including Venn diagrams, probability trees, and algebraic solutions.