A Course Outline for Geometry

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Geometry Course Outline

Content Area	**Formative Assessment Lessons**	**# of Days**
G0 Introduction and Construction G-CO Congruence 12, 13 ← G-CO 12,13 →		12
G1 Basic Definitions and Rigid Motion G-CO Congruence 1, 2, 3, 4, 5, 6, 7, 8 ← G-CO 6,7,8 → ← G-CO 1,2,3,4,5 →	Representing and Combining Transformations Analyzing Congruency Proofs	20
G2 Geometric Relationships and Properties G-CO Congruence 9, 10, 11 G-C Circles 3 ← G-CO 9,10,11 → ← G-C 3 →	Applying Angle Theorems Evaluating Statements About Length and Area Inscribing and Circumscribing Right Triangles	15
G3 Similarity G-SRT Similarity, Right Triangles, and Trigonometry 1, 2, 3, 4, 5 ← G-SRT 1,2,3,4,5 →	Geometry Problems: Circles and Triangles Proofs of the Pythagorean Theorem	20
M1 Geometric Modeling 1 G-MG Modeling with Geometry 1, 2, 3 ← G-MG 1,2,3 →	Solving Geometry Problems: Floodlights	7
G4 Coordinate Geometry G-GPE Expressing Geometric Properties with Equations 4, 5, 6, 7 ← G-GPE 4,5,6,7 →	Finding Equations of Parallel and Perpendicular Lines	15
G5 Circles and Conics G-C Circles 1, 2, 5 G-GPE Expressing Geometric Properties with Equations 1 ← G-C 1,2,5 → ← G-GPE 1 →	Equations of Circles 1 Equations of Circles 2 Sectors of Circles	15
G6 Geometric Measurements and Dimensions G-GMD 1, 3, 4 ← G-GMD 1,3,4 →	Evaluating Statements About Enlargements (2D & 3D) 2D Representations of 3D Objects	15
G7 Trigonometric Ratios G-SRT Similarity, Right Triangles, and Trigonometry 6, 7, 8 ← G-SRT 6,7,8 →	Calculating Volumes of Compound Objects	15
M2 Geometric Modeling 2 G-MG Modeling with Geometry 1, 2, 3 ← G-MG 1,2,3 →	Modeling: Rolling Cups	10
	TOTAL:	144

The Formative Assessment Lessons are available at http://map.mathshell.org/materials/lessons.php

High School Overview

Algebra 1	Geometry	Algebra 2
A0 Introduction	G0 Introduction and Construction	A0 Introduction
A1 Modeling With Functions	G1 Basic Definitions and Rigid Motion	A1 Exponential Functions
A2 Linear Functions	G2 Geometric Relationships and Properties	A2 Trigonometric Functions
A3 Linear Equations & Inequalities in One Variable	G3 Similarity	A3 Functions
Modeling Unit 1	M1 Geometric Modeling 1	Modeling Unit 1
A4 Linear Equations & Inequalities in Two Variables	G4 Coordinate Geometry	A4 Rational and Polynomial Expressions
A5 Quadratic Functions	G5 Circles and Conics	P1 Probability
A6 Quadratic Equations	G6 Geometric Measurements and Dimensions	S2 Statistics
S1 Statistics	G7 Trigonometric Ratios	Modeling Unit 2
Modeling Unit 2	M2 Geometric Modeling 2	
1. Make sense of problems and persevere in solving them.	4. Model with mathematics.	7. Look for and make use of structure.
2. Reason abstractly and quantitatively.	5. Use appropriate tools strategically.	8. Look for and express regularity in repeated reasoning.
3. Construct viable arguments and critique the reasoning of others.	6. Attend to precision.	

GO Intro and Construction
12 Days

In the *CCSS* for grades 6 and 7, students are asked to draw triangles and other polygons based on given measurements, and they begin to investigate relationships within the figures that are the basis for geometric postulates and theorems. In high school, students apply reasoning to complete geometric constructions and use properties of angles and segments to explain why the constructions work. They develop their visual recognition and vocabulary, which provides an essential foundation for further study of geometric theorems and relationships. By copying segments and angles and constructing congruent figures, the students deepen their understanding of congruence as it relates to the superposition principle (the placement of one object ideally in the position of another one in order to show that the two coincide). The focus here is not only on the motor skills and methods of construction, but on providing opportunities for students to deepen their understanding of the inherent properties that define geometric figures and the relationships between those properties.

Congruence G-CO

12. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*
13. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Teaching and Assessment Resources

http://map.mathshell.org/materials/index.php

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

G1 Basic Definitions and Rigid Motion
20 Days

The *CCSS* calls for students to define and understand congruence through the perspective of geometric transformations. Fundamental to this perspective are the rigid motions: translations, reflections, and rotations, all of which preserve distance and angle measure. Students explain the basis of rigid motions using geometric concepts, such as distance, movement along a parallel line, and movement along a circular arc with a specified center through a specified angle. They also build on their experience with rigid motions from earlier grades and deepen their understanding of what it means for two geometric figures to be congruent. Rigid motions and their properties are used to establish the criteria for triangle congruence, which are used later to prove other geometric theorems.

Congruence G-CO

1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.
6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Formative Assessment Lessons

Representing and Combining Transformations

This lesson is about transformations that preserve distance in a coordinate plane – translations, rotations, and reflections. These transformations are also called rigid motions or congruence transformations because they preserve the size and shape of geometric figures, although they may change the orientation of the figure in the plane.

In this lesson, students apply transformations to geometric images and describe transformations that carry one image onto another. In the collaborative activity, the students consider how to link two images with a transformation and explain their reasoning. Transformations can be performed by using the right “motion” along a particular path. In a translation, all points move vertically or horizontally for the given number of units. In a rotation, all points move in a circle about a fixed point (the circle’s center). In a reflection, points “flip” over the line of reflection along a line that's perpendicular to it. In a rotation and reflection, distances to the reference point or line must stay fixed.

When students begin linking images with transformations, they are (whether they realize it or not) working with transformations as functions and treating the images as inputs and outputs. In searching for all possible links that can be made, students re-use outputs as inputs and create sequences (compositions) of transformations. Students may be asked to look for sequences of two or more transformations that can be achieved with a single transformation. As they do this, students are actually checking for compositions of functions that are equivalent to other functions (for a given input).

The description of transformations using an algebraic rule arises out of performing them using the appropriate motion. In this lesson, using the algebraic rules that apply to the coordinates to *perform* the transformations is not a central focus.

Analyzing Congruency Proofs

In this lesson, students need to decide if two triangles sharing a given number of side lengths and a given number of angle measures must be congruent. First they are asked about three specific examples, and then they are provided with nine cards, each of them stating that two triangles share *n* side lengths and *m* angle measures, where *n* is 1, 2 or 3 and *m* is 0, 1 or 2. Hence they have nine cards (all combinations above are given). For each card, they need to decide if the triangles must be congruent or not. In the former case, they must prove it, and in the latter case they must find a counter example, and if possible, a further explanation of why the conditions are not the same as ASA or SAS conditions that do guarantee congruence. For this point, students must understand that the order of the corresponding sides and angles in ASA and SAS matters.

In order for students to access this lesson, they need to know some basic logic, like the fact that one counterexample is enough to establish the falsehood of a (universal) statement, while using inductive reasoning based on any number of confirming examples is not enough to prove such a statement. They also need to know the basic triangle congruence theorems, (SSS, SAS and ASA) and they need to know that SSA or AAS is not sufficient for congruence (although it IS sufficient in SSA if the angle is opposite the longer side, a result little known in school). Students also need to know similarity conditions, most crucially AA, and the fact that since all triangles share the same angle sum of 180 degrees, sharing two angle measures is equivalent to sharing all three of them.

Teaching and Assessment Resources

http://map.mathshell.org/materials/index.php

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

G2 Geometric Relationships and Properties
15 Days

Students use triangle congruence and their experience with geometric construction as a foundation for the development of formal proof. They explore various ways of writing proofs, including paragraph proofs, flow charts, two-column proofs, and labeled diagrams. Students are encouraged to focus on the validity of the underlying reasoning while they apply the different methods of formal proof. It is imperative that this unit include sufficient opportunities for discussion, reflection, student dialogue, and argument as students develop a deeper understanding of what constitutes a convincing proof. Through the practice of writing formal proofs, students extend their understanding and vocabulary with respect to the properties of lines, angles, triangles and quadrilaterals.

Congruence G-CO

9. Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints*.
10. Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point*.
11. Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals*.

Circles G-C

3. Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Formative Assessment Lessons

Applying Angle Theorems

In this lesson, students are given problems that require them to find interior and exterior angles of polygons. They are given several sample solutions to the problem which they are asked to analyze and criticize. In analyzing these sample solutions, students need to recognize whether false assumptions are being made, or if there are computational errors.

Students are given a sheet with relevant terminology and facts about angles formed by lines, angles formed by a transversal to parallel lines, the interior angle sum of a triangle, and the interior and exterior angle sums of polygons. In order to criticize some solutions, or to come up with their own, students need to be familiar with triangle congruence theorems and basic facts about parallelograms. They also may need to add auxiliary lines in a given figure.

Evaluating Statements about Length and Area

This lesson is about the perimeter and area of different kinds of figures: quadrilaterals, triangles, and circles. Students need to determine whether certain statements are always, sometimes, or never true, and prove their assertions. When the statements are always or never true, the proof entails providing an argument that applies to any figure of the kind being considered. When the statement is sometimes true, students need to provide examples, counterexamples, and, more importantly, the extent to which the statement is true. That is, they need to restrict the set of figures being considered so that the statement will always be true for those figures, and false for others.

In order to complete this lesson, students need to be familiar with the formulas for the perimeter and area of triangles, trapezoids, parallelograms and circles. They also need some experience proving and disproving geometric statements, as well as some experience with logical reasoning and proofs. They also need to know the congruence theorems for triangles, basic facts about the properties of parallelograms, and the fundamental theorem of similarity.

Inscribing and Circumscribing Right Triangles

This is a lesson about finding the radii of both the circumscribed and the inscribed circles of a right triangle. Students are asked to find them in a specific example, a 5-12-13 triangle that is not drawn to scale, and later they need to generalize their results to arbitrary right triangles. Besides the usual small group collaboration, they are given a few questions to reflect on how they did individual and collaborative work.

There are a few different ways to solve the problem. In order to do that, and to evaluate the sample student work that is given to students, they need to be familiar with the circle and triangle theorems of Geometry. Four circle theorems are provided, two of which are most relevant here: the one about the measure of an inscribed angle being half the measure of the corresponding central angle, and the one stating that tangent segments from a common point are congruent. Some of the triangle theorems and other supporting Geometry facts that students should know and be able to apply to this problem are the definition of distance between a point and a line; that a point on an angle bisector is equidistant from the sides of the angle; that the angle bisectors of a triangle are concurrent; that every triangle has a unique circumscribed circle (its center is the point of concurrency of the perpendicular bisectors of the sides); that every triangle has a unique inscribed circle (its center is the point of concurrency of the angle bisectors of its angles); and the formula for the area of a triangle given a side and the altitude from the opposite vertex.

Teaching and Assessment Resources

http://www.MathEducationPage.org/
http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
	A10: Floor Patterns	E09: Triangular Frameworks

G3 Similarity
20 Days

Students build on their experience with proportional reasoning and dilation from grades 7 and 8 to establish a formal understanding of similarity. As congruence is approached from the perspective of rigid motions, similarity is examined from the perspective of similarity transformations (dilations or rigid motions followed by dilations). Students investigate and verify the effects of dilation on the properties of parallelism and segment length, and relate these effects to the center and scale factor of dilation. They also identify criteria for the similarity of triangles, use the definition of similarity to determine if two figures are similar, and use similarity to solve problems. Students are expected to understand and explain similarity in figures from the perspective of similarity transformations, and to use the properties of similarity transformations to establish triangle similarity theorems.

Similarity, Right Triangles, and Trigonometry G-SRT

1. Verify experimentally the properties of dilations given by a center and a scale factor:

1a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.

1b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

2. Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

3. Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

4. Prove theorems about triangles. *Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity*.

5. Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Formative Assessment Lessons

Geometry Problems: Circles and Triangles

This Geometry lesson asks for the ratio of the areas of two equilateral triangles, one circumscribed and the other inscribed in the same circle. Then it asks for the ratio of the areas of two circles, one circumscribed and one inscribed in the same equilateral triangle.

There are many different ways to solve this problem. Students may use or derive facts about equilateral triangles, like the fact that their medians cut each other in segments of ratio 2 to 1. Depending on their approach, they may need some basic right-triangle trigonometry, or else knowledge of the ratio of side lengths of 30-60-90 triangles. They may need to know how to derive the area of an equilateral triangle given its side length, and they may need to know the formula for the area of a circle. They may also need to know about similarity and the effect on areas upon scaling by a given factor. They may use rotations and triangle congruence theorems. Finally, students are asked to examine carefully and criticize several sample solutions for correctness, clarity, and completeness.

Proofs of the Pythagorean Theorem

In this lesson, students need to use diagrams provided to them in order to prove the Pythagorean theorem in different ways. They also need to analyze, evaluate and correct different proofs that may have mistakes, given as sample student work. Together, they provide several different ways to prove the theorem.

Students need to be able to translate back and forth between the visual information of the geometric diagrams and the algebraic relationships that they reveal, through the consideration of lengths, areas, congruence and similarity. They also need to be able to explain why a geometric figure really is what it looks like in a diagram (for example, a square or a trapezoid) by considering lengths and angles. They have to be able to label lengths of segments in a useful way, and to visualize how one diagram can be changed into another by moving around congruent pieces, in order to represent certain quantities in two different ways.

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
		E04: Proofs of the Pythagorean Theorem
		E08: Pythagorean Triples
		E15: Circles and Squares

M1 Geometric Modeling 1
7 Days

This unit is intended to provide students with an opportunity to apply geometric concepts in modeling situations. Students choose and use appropriate geometric tools and concepts to analyze empirical situations. Modeling allows the students to understand the situation better and to improve decision-making and interpretation. This could include using geometric shapes to represent objects, using geometric concepts to represent situations, or using geometric methods to solve design problems.

Modeling with Geometry G-MG
1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Formative Assessment Lessons

Solving Geometry Problems: *Floodlights*

In this lesson, students are asked to solve a geometric problem given as a word problem. A person stands between two floodlights that are at the same height. Students need to find the length of the shadows and hence the total length of the two shadows. Then the person starts moving toward one of the floodlights, and students are asked to explain what happens to this total length.

First, students need to be able to model this situation with an appropriate geometric picture. They also need to recognize that the key to the problem (that the length of the combined shadows remains constant) is similar triangles, and once they find useful similar triangles, they need to use the facts about ratios of corresponding sides being equal. In order to establish the similarity of the triangles, students need to know that opposite sides of a rectangle are parallel, that alternate interior angles cut by a transversal to two parallel lines are equal, and that triangles sharing two angle measures are similar.

Teaching and Assessment Resources

http://map.mathshell.org/materials/index.php

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

G4 Coordinate Geometry
15 Days

Students use the coordinate plane and algebra to verify geometric relationships. This is an extremely important unit, as it provides students with the opportunity to refine their understanding of basic geometric theorems through empirical investigation, experiment, and algebraic proof. Applications of slope extend and strengthen student experiences from 9th grade, and deepen conceptual understanding of parallel and perpendicular lines and the properties of quadrilaterals. Students build on their work with the Pythagorean theorem from 8th grade by using distance to investigate geometric figures and solve problems.

Expressing Geometric Properties with Equations G-GPE

4. Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*
5. Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
7. Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula.

Formative Assessment Lessons

Finding Equations of Parallel and Perpendicular Lines

In this lesson, students work with sets of linear equations written in different forms to sort out pairs of parallel and perpendicular lines from other pairs of intersecting lines. Pairs of lines that are coincident are not considered here.

In the assessment, students sort out the pairs of parallel and perpendicular lines in the process of looking for four equations whose graphs intersect to form a rectangle (as it turns out, two of the vertices occur on the x- and y-axes). In the lesson, students sort pairs of equations into categories based on whether their lines are parallel, perpendicular, or intersect at a particular point (y-intercept, x-intercept, or another point).

In their search, students will use the equations of the lines as a starting point for figuring out how they relate to each other. Students who rely solely on the way the equation is written to make decisions may or may not be successful. After all, there is no guarantee that pairs of equations *written in a particular form* will have the relationship the student is looking for, unless the form is one that reveals key information about the lines. Slope-intercept form is especially useful for spotting parallel and perpendicular lines, while other forms may prove more useful for deciding if and how lines intersect.

Therefore, the mathematical focus of this lesson is not on the *skill* of "converting to slope-intercept form to classify pairs of linear equations", but rather on the recognition of the usefulness of forms of equations that reveal information. Students may use all connections between representations of lines (algebraic, graphical, and numerical) that they find helpful for determining the relationships between the linear equations. For instance, it is not necessary to change the form of an equation to decide if the line contains a desired intersection point.

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
	A22: Square	

G5 Circles and Conics
15 Days

In this unit, students prove and apply theorems about circles. They also examine how central, inscribed and circumscribed angles relate to chords, radii, and arcs of the circle. The idea of radian measure is introduced through the idea that all circles are similar, and the fact that the similarity of all sectors with the same central angle shows that arc length is proportional to the radius. By using similarity as the basis of this topic, students will have a more complete conceptual framework upon which to place the idea of radian measure, and a more reliable way to reconstruct the idea. Students also extend their thinking about the Pythagorean theorem and coordinate distance to write the equation of a circle given the center and radius, and to graph a circle given its equation. Students are expected to translate between the geometric description and the equation of a circle and a parabola.

Circles G-C

1. Prove that all circles are similar.
2. Identify and describe relationships among inscribed angles, radii, and chords. *Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.*
5. Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector

Expressing Geometric Properties with Equations G-GPE

1. Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Formative Assessment Lessons

Equations of Circles 1

In this lesson, students need to find an equation for a circle given some information about its graph, such as the coordinates of the center and the radius. They also need to be able to graph a circle given its equation, and find the possible coordinates of a point on the circle given the other coordinate. The equations are all given (with only minor changes like the order of the terms) in the form where the squares have been completed, so that the center and radius are immediately apparent from the equation. Students are given several such equations, and they have to correctly place each one in a matrix of different center locations and radii. In order to understand the lesson and find points on circles, as well as the equations, students need to know the Pythagorean theorem and the distance formula, be familiar with the Cartesian (coordinate) plane, and be able to manipulate square roots.

Equations of Circles 2

This lesson is also about circles and is similar to *Equations of Circles 1*. Roughly the same skills, knowledge and ability are necessary. Again the equations given or asked for are already in completed squares. In this lesson, students need to find the *x*- and *y*- intercepts (if any) of circles. They are then given several equations and graphs, they need to place each in a chart that is a matrix with the number of x- and y- intercepts that a circle may have (0, 1, or 2 of each). They also need to find equations for circles having certain properties and a given number of x- and y- intercepts, and they have to explain why the equations they come up with produce circles with the required properties. Another task of the lesson involves considering three circles and finding different criteria under which any one of them could be considered the odd one out.

Sectors of Circles

In this lesson, students study sectors of circles, paying attention to their angle, radius, arc length, and area. Students are led through a discussion of radians, arc length, and area of sectors.

Students need to find the formulas for arc length and area by understanding how the angle of the sector determines the fraction of a circle that it makes up, and then use proportional reasoning together with the formulas for the circumference and area of a circle. Given three concentric circles with specified radii (in the ratio 1:2:4), they need to find different sectors on any of the circles that have the same area or arc length. For this, they need to understand what happens to lengths and areas when scaling figures. They also need to understand, using the formulas they obtained, how the arc length and area of a sector change with its angle and with its radius.

Students are then asked to sort domino cards into a loop, where half of each card is a sector (whose arc length, radius, perimeter and area they need to compute given one of these quantities) and the other half is a set of instructions for changing or maintaining some of these quantities. They also need to match cards that describe changes in arc length or area with cards that describe changes in angle and radius to produce the changes given in the first set.

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N10: Expressing Geometric Properties with Equations	A13: Temple Geometry	

G6 Geometric Measurements and Dimensions
15 Days

The *CCSS* requires students in the middle grades to know and apply formulas for the circumference and area of a circle and the volume of prisms, cones, cylinders, and spheres. In this course, as students continue to solve problems using these formulas, they are also asked to explain them using informal arguments and to think more deeply about the mathematics that these formulas express. This may include investigations on how area and volume scale under similarity transformations, or how area formulas are extended into three dimensions to create volume formulas. Students build on their experience of standard 7.G.3, where they described the two-dimensional figures that result from taking cross sections of three dimensional figures, by identifying cross sections of more complex three-dimensional figures and three-dimensional objects created by the movement of two-dimensional figures. This type of abstract thinking about the relationship between two- and three-dimensional space is a significant cognitive demand for students, but is critical to the understanding of many key concepts in this course.

Geometric Measurement and Dimension G-GMD

1. Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. *Use dissection arguments, Cavalieri's principle, and informal limit arguments.*
3. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
4. Identify the shapes of two-dimensional cross-sections of three dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

Formative Assessment Lessons

Evaluating Statements about Enlargements (2D & 3D)

This lesson is about the change in perimeter, area, and volume resulting from scaling two-dimensional and three-dimensional figures. Several statements are made about the change in perimeter, area, or volume of figures resulting from applying a scale factor to a linear dimension of the figure. The 2-D figures are rectangles and circles, and the 3-D figures are rectangular prisms, spheres, right cylinders, and cones. The required formulas for the volume and lateral area of solids are given to the students, as are the formulas for the circumference and area of a circle.

Students need to decide, for each given statement, if it is true or false. They need to explain their reasoning to each other, and if a statement is false, they are asked to change it so as to make it true.

The purpose of the lesson is for students to understand that if two similar figures are related by a linear scalar factor k, then corresponding linear measurements (like perimeter) will be related by the same factor k, while corresponding area measurements (like surface area or lateral area) will be related by a factor k^2 and corresponding volumes will be related by a factor k^3. But students are also asked to consider what happens to perimeters, areas and volumes if some dimensions are scaled, but not all (breaking similarity).

2D Representations of 3D Objects

This lesson is about the visualization of planar cross sections of three-dimensional objects. The 3D objects themselves are not provided physically, but as 2D drawings that represent them. Students therefore need to visualize the 3D objects from the drawings, and then also visualize planar cross sections of the objects. Some of the 3D objects represented by drawings are spheres, hemispheres, cones, tetrahedrons, octahedrons, cubes, rectangular and triangular prisms, rectangular pyramids, and cylinders.

Students need to consider what the surface of the water would look like as a 3D object, placed in a certain rigid orientation, is being drained. They need to draw the cross sections at several distinct moments. The orientation of the object matters (in most cases—not for a sphere), and students need to realize this.

Later, students are provided with cards that they need to match. Six of the cards contain drawings of two identical 3D objects, one on top of the other, positioned with different orientations, and connected with a thin vertical pipe. In each case, the top object is initially filled with water and starts draining into a lower, identical object. Twelve cards contain drawings with possible sequences of cross sectional shapes for either the top or the bottom object in the six cards. Students need to map each of the twelve sequences to its corresponding object, as it is being drained (top) or filled (bottom). Some of the cross section cards are missing drawings in the sequence, and students need to supply the missing cross sections. They need to explain their drawings and their matching.

This lesson requires a good deal of visualization, communication, and justification skills, as well as drawing ability. Students should also know the basic properties of common 3D and 2D geometric figures and be able to use them in their argumentation. Finally, they need to understand what is meant by a planar cross section, i.e., the intersection of the 3D object with a plane (in this lesson, a horizontal plane), and they must know that two non-parallel planes intersect along a line. A useful fact for the lesson is that if a plane intersects two parallel planes, the resulting lines are parallel. Students can apply this fact to deduce properties about the possible cross sections of prisms (for example, cubes).

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
	A03: Funsize Cans	E03: Fruit Boxes
	A11: Glasses	E12: Bestsize Cans
		E16: Propane Tanks

G7 Trigonometric Ratios
15 Days

Students apply their knowledge of similarity to right triangles, and understand that trigonometric ratios in right triangles extend across all similar triangles and are therefore properties of the angles. The idea that trigonometric functions are properties of angles and depend on angle measure is a critical piece of understanding, and provides a basis for all further work in trigonometry. Students use the trigonometric ratios, the Pythagorean theorem, and properties of right triangles to investigate relationships in figures and solve problems.

Similarity, Right Triangles, and Trigonometry G-SRT

6. Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
7. Explain and use the relationship between the sine and cosine of complementary angles.
8. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.w

Formative Assessment Lessons

Calculating Volumes of Compound Objects

In this lesson, students need to find the volume of different-shaped glasses by decomposing them, if necessary, into simpler shapes whose disjoint union makes up the whole glass. These shapes are cylinders, hemispheres, and cones. Students are given the necessary volume formulas. Students are also asked to find the height of the water in some of these glasses when the glasses are half full. This involves more demanding reasoning and computation. In the case of a cone, students need to make use of similar right triangles and proportional reasoning to obtain a relationship between the radius of the surface of the water and the height of the water in the glass.

Students are also asked to criticize and correct sample solutions. Finally, one multiple-choice question asks students to think about dimensional analysis in order to choose a correct volume formula, and to consider whether it makes sense for a given height or a given diameter to be squared in that formula.

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
	A5: Hopewell Geometry	

M2 Geometric Modeling 2
10 Days

This unit provides students with a further opportunity to apply geometric concepts, use appropriate geometric tools, and analyze empirical situations through modeling. They draw on learning from units G3, M1, and G6 to strengthen and deepen conceptual understanding through application.

Modeling with Geometry G-MG

1. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
2. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).
3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Formative Assessment Lessons

Modeling: *Rolling Cups*

This is a lesson in which students need to analyze videos and data, make their own hands-on and computer-based experiments, and make a mathematical model, in order to figure out the roll radius of any cup (in the shape of a frustum) given its wide diameter, narrow diameter, and slant height. Students need to think about the relationship between the roll radius and the three given measures. They need to discover the idea of fixing two of these three measures at a time and varying the third, and explain what happens in these cases. They also need to reason that if the wide and narrow diameter are very close to each other, the roll radius is large, and in the limiting case when the diameters are equal, there is an "infinite" roll radius and the cup rolls in a straight line rather than a circle.

A complete solution requires drawing the vertical cross section of a cup and extending this drawing to a cone, in order to use similar triangles and the resulting proportional relationships in the figure. The roll radius, which is the slant height of this cone, can then be obtained in terms of the three given measures of the cup. Once students have such an expression for the roll radius, they can study algebraically how it varies with each of the measures, and they need to confirm that these results agree with their experimental ones.

Teaching and Assessment Resources

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

http://map.mathshell.org/materials/index.php