A Course Outline for Algebra 1

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for the Bill & Melinda Gates Foundation

January 2014

Algebra 1 Course Outline

Content Area	Formative Assessment Lessons	# of Days
A0 Introduction: Mathematical Investigations		5
A1 Modeling with Functions N-RN The Real Number System 3 N-Q Quantities 1, 2, 3. F-IF Interpreting Functions 1, 2, 3, 4, 5, 9 F-LE Linear, Quadratic and Exponential Models 1, 2, 3, 5 ← F-IF 1,2,3,4,5 → ← N-Q 1,2,3; F-IF 9; F-LE 1,2,3,5 → ← N-RN 3 →	Functions and Everyday Situations Modeling: Having Kittens Rational and Irrational Numbers 1	15
A2 Linear Functions F-IF Interpreting Functions 6, 7a, 9 F-BF Building Functions 1a F-LE Linear, Quadratic and Exponential Models 1 ← F-IF 6 → ← F-IF 7a,9; F-BF 1a; F-LE 1 →	Comparing Investments Generalizing Patterns: Table Tiles	16
A3 Linear Equations & Inequalities in One Variable A-APR Arithmetic with Polynomials and Rational Expressions 1, 3 A-SSE Seeing Structure in Expressions 1, 2, 3a, 3b, 3c A-REI Reasoning with Equations and Inequalities 1, 3, 11 A-CED Creating Equations 1, 3, 4 ← A-SSE 1,2; A-APR 1; A-CED 1,3,4; A-REI 1,3,11 → ← A-SSE 3; A-APR 3 →	Manipulating Polynomials Interpreting Algebraic Expressions Sorting Equations and Identities Building and Solving Equations 2	16
A4 Linear Equations & Inequalities in Two Variables A-CED Creating Equations 2, 3, 4 A-REI Reasoning with Equations and Inequalities 5, 6, 10, 12 ← A-CED 2,3,4; A-REI 10,12 → ← A-REI 5,6 →	Solving Linear Equations in Two Variables Optimization Problems: Boomerangs Defining Regions Using Inequalities	16
A5 Quadratic Functions F-IF Interpreting Functions 4, 5, 6, 7a, 7b, 8a, 9 F-BF Building Functions 1a, 3 ← F-IF 4,5,6 → ← F-IF 7a,7b,8a,9; F-BF 1a → ← F-BF 3 →	Forming Quadratics	20
A6 Quadratic Equations A-SSE Seeing Structure in Expressions 3a, 3b A-REI Reasoning with Equations and Inequalities 4a, 4b ← A-REI 4a,4b → ← A-SSE 3a,3b →	Solving Quadratic Equations: Cutting Corners	26
S1 Statistics S-ID Interpreting Categorical and Quantitative Data 1, 2, 3, 4, 5, 6, 7, 8, 9 ← S-ID 7,8,9 → ← S-ID 5,6 → ← S-ID 1,2,3,4 →	Representing Data 1: Using Frequency Graphs Representing Data 2: Using Box Plots Interpreting Statistics: A Case of Muddying the Waters Devising a Measure for Correlation	30
Total days		144

Most of the Formative Assessment Lessons are available at http://map.mathshell.org/materials/lessons.php

High School Overview

Algebra 1	Geometry	Algebra 2
A0 Introduction	G0 Introduction and Construction	A0 Introduction
A1 Modeling With Functions	G1 Basic Definitions and Rigid Motion	A1 Exponential Functions
A2 Linear Functions	G2 Geometric Relationships and Properties	A2 Trigonometric Functions
A3 Linear Equations & Inequalities in One Variable	G3 Similarity	A3 Functions
Modeling Unit 1	M1 Geometric Modeling 1	Modeling Unit 1
A4 Linear Equations & Inequalities in Two Variables	G4 Coordinate Geometry	A4 Rational and Polynomial Expressions
A5 Quadratic Functions	G5 Circles and Conics	P1 Probability
A6 Quadratic Equations	G6 Geometric Measurements and Dimensions	S2 Statistics
S1 Statistics	G7 Trigonometric Ratios	Modeling Unit 2
Modeling Unit 2	M2 Geometric Modeling 2	
1. Make sense of problems and persevere in solving them.	4. Model with mathematics.	7. Look for and make use of structure.
2. Reason abstractly and quantitatively.	5. Use appropriate tools strategically.	8. Look for and express regularity in repeated reasoning.
3. Construct viable arguments and critique the reasoning of others.	6. Attend to precision.	

A0 Mathematical Investigations
5 Days

This introductory unit is designed to provide students with an opportunity to engage in worthwhile mathematical investigations; it has a number of related purposes. The most important of these is to let students draw heavily on the eight Mathematical Practices and establish their importance to this entire course. Through working with the mathematical practices, students will come to understanding the importance of seeing, doing, re-constructing, and supposing, in learning mathematics, and hopefully also realize that mathematics is not just facts to be memorized.

Examples of the kinds of mathematical investigations that students will work on during this unit can be found at Henri Picciotto's Mathematics Education Page (http://www.MathEducationPage.org/).

One of these investigations is called "Area on Graph Paper"; it explores the least and greatest perimeters that can be made with polyominoes of a fixed area. This calls on students to make and test hypotheses, be systematic, and draw conclusions. It will encourage students to deploy MP.1 (*Make sense of problems and persevere in solving them*) and MP.7 (*Look for and make use of structure*). (http://www.MathEducationPage.org/new-algebra/new-algebra.html)

The McNuggets Problem, another type of mathematical investigation, which can also be found on Henri's website in the document "Early Mathematics: A Proposal for New Directions" (http://www.mathedpage.org/early-math/early.html) is particularly valuable in an introductory unit such as this. Students are invited to consider the number of chicken nuggets that might be bought if a fast food store sells them in quantities of 6, 9, and 20. Students are asked to determine how many nuggets can and cannot be bought. Students are also invited to determine what is the largest number of nuggets that cannot be bought, and then prove that every number greater than this largest number can be bought. (http://www.MathEducationPage.org/early-math/early.html)

Another interesting investigation, titled "Which is Greater — Using Linear Functions," provides students with a number of linear equations written in slope intercept form. Students are asked to determine which of these would produce graphs that intersect to the left of the y-axis, to the right of the y-axis, or on the y-axis, or not intersect at all. Then students are asked to say how they might determine each of these cases without graphing. This genre of investigation will call upon students to put to work their learning from Grade 8. (http://www.MathEducationPage.org/calculator/which-is/greater-linear.pdf)

Another purpose of this introductory unit is to establish the norms for doing mathematics, and set the scene for the rest of the course. Teachers will be encouraged to enact these investigations in a way that enables students to take responsibility for their own learning and act as instructional resources for each other. The teacher will provide feedback that moves the learning forward and engineer effective discussion so as to facilitate learning. The intent of each mathematical investigation will be made explicit so as to provide a purpose for students.

The genius of this type of investigation is that, in a heterogeneous class of students, it is highly likely that every student will be able to do something, but also highly likely that no student will be able to do everything.

A1 Modeling with Functions
15 Days

Using functions to describe relationships between quantities is a core idea in high school algebra and mathematical thinking in general. Students began working with linear functions in grade 8, and in this unit they expand that work to address a variety of types of situations that can be modeled using functions, including exponential and quadratic relationships. Detailed analysis of specific types of functions will be addressed in subsequent units of study. The work here is more focused on appropriate modeling, understanding of domain and range, using function notation, and interpreting functions in multiple representations. There is a particular emphasis on the differences between linear and exponential models, and specifically the types of growth these functions produce.

In the *CCSS*, students in grades 4 and 5 investigate, interpret, and relate various units of measure within the domain Measurement and Data. In the middle grades, they apply their experience with unit measurement to understand proportional relationships between quantities expressed in different units. In this course, students identify appropriate and relevant units to help them understand and solve problems. Their work with quantities and the relationships between them provides context and meaning for further work with expressions, equations, and functions.

The Real Number System N-RN

3. Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities N-Q

Reason quantitatively and use units to solve problems

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
2. Define appropriate quantities for the purpose of descriptive modeling.
3. Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

Interpreting Functions F-IF

1. Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.
2. Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.
4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

Linear and Exponential Models F-LE

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.
1a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
1b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
1c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.
2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
3. Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.
5. Interpret the parameters in a linear or exponential function in terms of a context.

Formative Assessment Lessons

Functions and Everyday Situations

This lesson is about being able to model everyday situations using graphical and algebraic representations of real-valued functions. Students need to translate between the situations described, the graphs, and the algebraic formulas for the functions that serve as models.

In the first four and the last four situations presented to them, students need to produce appropriate graphs that show the relationship between the two quantities being described. They need to consider whether each graph should be discrete or continuous based on the context. Then they are given formulas for the functions that model the situations, and they need to match each to its corresponding situation and graph. Finally, they are asked several questions requiring them to interpret the graphs in the context of the situation described.

There are also twelve “everyday situation” cards that students need to match with twelve graph cards, and later on each pair of cards needs to be matched to an equation card.

There are several functions used to model the many situations of this lesson: linear (through the origin and otherwise), quadratic, exponential, rational, radical, trigonometric. Some contexts are discrete, some are continuous, and some are step functions with one variable discrete and the other continuous. Students need to have a working knowledge of how to handle a variety of such functions, their graphs, their domains, and their properties. They also need to know how these properties make them good models for certain situations. Finally, they need to know or figure out a few facts about the everyday situations presented to them, for instance that the radius of a sphere will grow as the cube root of its volume, or the laws of physics governing free falling objects thrown with a given initial velocity, or that in some situations the product of the quantities described must be a constant.

Modeling: Having Kittens

This lesson is a very rich modeling task. A claim is made about the possible number of descendants of a (female) cat in an 18-month period. Students need to decide whether or not the claim is reasonable. They are given some information, like the average number of litters a female has in one year, the average time of pregnancy, the average size of a litter, and the age range over which a female cat can have kittens.

Students need to decide what pieces of information to use, and they need to explicitly state the assumptions that they will make in order to model the situation and figure out an estimate for the number of descendants. For example, they may make the assumption that all kittens in the model are female, in order to avoid complications about descendants of male cats, or they may assume otherwise and deal with descendants of males. Students need to keep track of the litters of every new generation, and they need to realize the exponential nature of the growth. They need to make posters with complete explanations and diagrams, and they need to explain it to others and to critique others' posters, noting clarity, completeness, strengths, and weaknesses of the different solutions. They also need to critique three given sample solutions, each with different flaws (for example, counting only the kittens born from the original cat). If they choose to model the problem algebraically, they will need some familiarity with exponential functions.

Rational and Irrational Numbers 1

In this lesson, students need to classify several real numbers given in different forms (like fractions, decimals, repeating decimals, square roots, and arithmetic combinations of some of these), as rational or irrational. In some cases the notation or description of the number does not provide enough information to make the classification, and students need to discover these cases.

In order to succeed, they need to know the definition of rational (and irrational) number. They also need to have an agreement as to what a "fraction" is, especially that a "fraction," as used here, is not equivalent to a rational number. Finally, they need to know for a fact that the square root of an integer that is not a perfect square is an irrational number. They are not required to prove this. They should know that the rational numbers are closed under the arithmetic operations, and be able to use this fact (via an easy contradiction argument) to establish results such as that the sum of a rational and an irrational number must be irrational. They need to be able to manipulate terminating and repeating decimals and write them as a quotient of two integers. But they are not required to formally explain the meaning of the repeating decimals' notation or the arithmetic manipulations performed on them. That is to say, they are not asked to deal with infinite series. Finally, they need to simplify products or sums in order to establish the rationality or irrationality of some numbers.

Students are given several numbers and categories of decimals to place them under. These categories are: terminating decimals, repeating non-terminating decimals, and non-repeating non-terminating decimals. They need to realize that terminating and repeating decimals are always rational, while non-terminating and non-repeating decimals are always irrational.

Assessment Resources

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N06: Interpreting Functions	A16: Sorting Functions	
	A19: Leaky Faucet	
	A24: Yogurt	

A2 Linear Functions
16 Days

This unit continues the work from unit A1, but with a particular focus on linear functions. Students continue to understand functions as mathematical relationships as they investigate rates of change and the key features of graphic representations and relate them to the specific relationship being modeled. Students write functions that describe such relationships using arithmetic and geometric sequences. The emphasis in this unit is not so much on the algebraic expression of linear functions, but on the conceptual understanding of linear functions as mathematical relationships and the specific character of such relationships as compared to other types of function.

Interpreting Functions F-IF
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
7a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum*.

Building Functions F-BF
1a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Linear, Quadratic and Exponential Models F-LE
1. Distinguish between situations that can be modeled with linear functions and with exponential functions.
1a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.
1b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
1c Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Formative Assessment Lessons

Comparing Investments

This lesson is about investments with simple and compound interest rates. The main mathematical tasks are to compare the linear versus exponential models that arise from these investments, and to be able to represent and describe them in a variety of ways, including verbal descriptions, algebraic equations, tables of values, and graphs.

Students are asked to compare a simple interest situation (linear) with a compound interest situation (exponential). They need to find equations for each, and match the situations to the appropriate graph among a few that are provided. They also need to compare investments of each type over shorter and longer time intervals. By identifying which among three simple interest situations is the “odd one out,” students investigate the features of each investment plan and how they affect the investment over time, as well as how they turn up in the equations.

They need to do the same with three compound investment situations. After this, they are required to match verbal descriptions with equations, tables and graphs. Finally, they are asked to match these with additional descriptions comparing the investments in terms of the time they take to double the initial amount of money or how much money they yield over different time intervals. In order to engage in this lesson, students need to be familiar with linear and exponential functions, their properties, and their graphs.

Generalizing Patterns: Table Tiles

Through the context of tiling square tabletops of different sizes, students need to discover, generalize, and prove the correctness of three sequential patterns. One of the patterns yields a constant sequence, one yields a linear (arithmetic) sequence, and the third one yields a pattern described explicitly by a quadratic function.

Students may obtain recursive formulas to discover the way in which the first and second consecutive differences behave. If they do, they need to justify the recursive formulas, and, more importantly, they need to use them in order to find explicit formulas that generalize their results for tables of arbitrary size. Students need to understand the difference between a conjecture and a formal argument that proves the conjecture.

They need to find and organize data, make predictions or conjectures, and prove them. Students are also asked to criticize sample solutions for correctness and completeness of the arguments. This lesson demands some familiarity with linear and quadratic equations, and with the behavior of first and second consecutive differences in tables of values coming from linear or quadratic functions evaluated at consecutive integers.

Assessment Resources

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N08: Linear and Exponential Models	A02: Patchwork	E14: Best Buy Tickets
	A20: Multiplying Cells	

A3 Linear Equations & Inequalities in One Variable
16 Days

This unit focuses on the algebra of equations and inequalities in one variable. Students deepen their understanding of the properties of equality and how the basic mathematical operations can be used to transform expressions, equations, and inequalities. These understandings form a basis for students to solve equations and inequalities in one variable and to rearrange formulas to isolate a specific quantity. By looking deeply into the meaning of equations and what it means to solve them, students use algebraic tools and concepts to investigate and interpret equations in a useful way. Students cannot effectively learn how to solve equations and inequalities by using a purely algorithmic or stepwise approach. It is imperative that they grapple with the mathematic principles that make the solving process work. Inequalities in one variable are presented as unique mathematical objects with infinite solution sets that can be expressed using a number line.

Seeing Structure in Expressions A-SSE
1. Interpret expressions that represent a quantity in terms of its context.
1a. Interpret parts of an expression, such as terms, factors, and coefficients.
1b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, Interpret $P(1+r)^n$ as the product of P and a factor not depending on P.*
2. Use the structure of an expression to identify ways to rewrite it. *For example, see x^4-y^4 as $(x^2)^2-(y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2-y^2)(x^2+y^2)$*
3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
3a. Factor a quadratic expression to reveal the zeros of the function it defines.
3b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
3c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{\frac{1}{12}})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Arithmetic with Polynomials and Rational Expressions A-APR
1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Reasoning with Equations and Inequalities A-REI
1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

Creating Equations A-CED

1. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*.
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R*.

Formative Assessment Lessons

Manipulating Polynomials

In this lesson, students are given the first few terms of several sequences of rectangular arrays with patterns of black and white dots. They need to recognize and extend the patterns, and find algebraic expressions for the number of black dots, the number of white dots, and the total number of dots in the nth term of each sequence. They also need to verify that the sum of the number of black and white dots equals the total number of dots in the nth array. Students are also given several cards with diagrams of arrays and several cards with algebraic expressions for the number of dots (black, white, total), and they need to match them and explain their reasoning. Finally, they need to create a few patterns of their own and their corresponding expressions.

As the patterns involved are all linear or quadratic, students need to be able to recognize such patterns and represent them algebraically as expressions, as well as manipulate these expressions formally to verify that the sums are correct. They must also be able to visualize these expressions as geometric patterns in order to go back and forth between the diagrams and the algebraic expressions.

Interpreting Algebraic Expressions

In this lesson, students need to write a one-variable expression from a description of the expression given in words. They are then shown four sample equations, and asked to determine if the expressions on both sides are actually equivalent, and to replace one if they are not. They also need to match four sets of cards for linear and quadratic expressions in one variable: symbolic expressions, word descriptions, tables of values, and area models. Students need to be familiar with the distributive property and order of operations, and to use them to recognize equivalent expressions.

Sorting Equations and Identities

In this lesson, students are asked to both write and classify equations based on the number of solutions in the solution set. Students will write equations that have no solutions, some solutions (one or two), and infinite solutions. Students will also classify equations into sets that are never true, sometimes true, or always true. Except for the case where the solution set is an infinite but proper subset of numbers, these classification systems are two ways of saying the same thing. Students will develop the notion of an "identity" as an equation that is always true.

The families of equations that students will work with are primarily linear and quadratic, with a few variations such as an equation in two variables and a proportion. Some linear and quadratic equations are chosen to highlight common mistakes students make when applying the distributive property, including multiplying binomials; however, these skills are not the intended focus of the lesson. The logic that students apply to categorize the equations is more central.

A variety of methods may be used to classify equations, so the approaches a student uses may vary by equation. For example, a student may choose to use substitution to reason that an equation is “sometimes true” by showing it has both solutions and non-solutions. Since classifying is the mathematical focus of the lesson, any correct method that occurs to the student is acceptable.

Note: Students who have studied imaginary numbers can be expected to use that knowledge here as the answer key is easily adjusted.

Building and Solving Equations 2

This is a lesson about solving linear equations in one variable. The variable can appear on one or both sides of the equation. Students are required to record and explain each step, and once they obtain solutions, they need to check them by reversing the steps of the solution to obtain the original equation, as well as substituting the solution value into the equation.

In order to succeed, students need to know and apply the order of operations and the distributive law. They also need to understand and use (in order to simplify the equations) the fact that adding or subtracting the same number to both sides of an equation results in a logically equivalent equation, as does multiplying/dividing both sides by the same nonzero number.

Assessment Resources

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
	A15: Multiple Solutions	

A4 Linear Equations & Inequalities in Two Variables
16 Days

Students build on algebraic skills highlighted in unit A3 to create, interpret, and graph equations and inequalities in two unknowns. Particular attention is placed on the idea that the graph of a linear equation in two unknowns represents the set of all its solutions plotted on the coordinate plane. Inequalities in two variables are presented as unique mathematical objects that have a solution set, which can be described by a half-plane when graphed. Students apply their understanding of equations and inequalities in two unknowns and their graphical representations to work with systems of equations and inequalities. In 8th grade, students solved simultaneous linear equations both graphically and algebraically, and they build on that work here by focusing on justification of the methods used and on a deeper analysis of systems with no solution or infinite solutions. They also solve systems of inequalities by identifying the intersection of the two solution sets, shown graphically as half-planes.

Creating Equations A-CED

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*
4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*

Reasoning with Equations and Inequalities A-REI

5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Formative Assessment Lessons

Solving Linear Equations in Two Variables

The goal of this lesson is for students to gain an understanding of how linear equations in two variables can be used to model real-life situations. In the lessons, students will interpret the meanings of linear equations in two variables in different contexts. They must also critically analyze incorrect interpretations of linear equations and explain why the interpretations are wrong. They will use and analyze several methods in the course of the lesson: elimination, substitution, guess and check and graphing the equations to find the intersection.

Optimization Problems: Boomerangs

This is a lesson about optimization. Students need to maximize a profit, which can be represented by a linear function of two variables. The constraints on the variables yield a feasible region bounded by the axes and two linear equations, resulting in a convex quadrilateral in the first quadrant. Students may graph the feasible region, which involves solving a system of linear inequalities in two variables. Once they have the feasible region, they may test the lattice points contained in it to find the maximum profit (the context makes the problem discrete, so only lattice points can be allowed as solutions). They may know or discover that in fact the maximum must be attained at a vertex of the feasible region, if the vertices are all lattice points, which is the case here. Students may instead organize data on a table to try to find the maximum profit.

Students are also given several sample responses, each with different strengths and weaknesses, and asked to analyze and critique them. To succeed in this lesson, students must apply what they know about linear equations to model a realistic situation, representing the constraints and variables mathematically.

Defining Regions using Inequalities

This lesson is about systems of linear inequalities in two variables. Students need to substitute values for *x* and *y* in the inequalities to find out if certain points belong to the solution set, i.e., the region (or lattice points in the region, since the solutions are lattice points by convention) where all the inequalities are simultaneously satisfied. They need to be able to graph linear equations and inequalities, and graph the region that forms the solution set of a system of inequalities. They also need to determine the usefulness of a certain clue, given a number of previous clues, and understand that some clues may be superfluous, while the most useful clues are the ones that restrict the solution set the most. Finally, students use a prescribed list of inequalities to construct a problem with a unique lattice point as its solution.

Assessment Resources

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N04: Creating Equations		
N05: Reasoning with Equations and Inequalities		

A5 Quadratic Functions
20 Days

Students build on their experience with linear and exponential relationships as they investigate quadratic functions. They compare quadratic functions to linear and exponential functions and identify relationships in which the quadratic function is the most appropriate model. Continuing their work with domain and range, students graph quadratic functions and identify key features of the graphs. This includes understanding how different algebraic forms of quadratic functions can be used to identify key features of the graph, such as the roots, the vertex, and whether the function has a maximum or a minimum.

Interpreting Functions F-IF

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★
5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*
6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

7a. Graph linear and quadratic functions and show intercepts, maxima, and minima.
7b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.
8a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

1a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Formative Assessment Lessons

Forming Quadratics

The goal of this lesson is for students to understand the different types of equations that describe quadratic functions, and the information that the different equations readily supply. In the standard form, the constant term gives the y-intercept, and the sign of the leading coefficient determines whether the function has a minimum (the graph is a parabola whose vertex is its lowest point) or a maximum (the graph is a parabola whose vertex is its highest point). In the factored form (factoring over the reals), the factors give the (real) roots, if there are any. These are the x-intercepts of the graph. A squared linear factor means that there is a double root, and the vertex of the parabola lies on the x-axis. The vertex form gives the coordinates of the vertex, and can also be easily used to decide if the function has real roots or not, because it shows whether the function has a minimum or a maximum, and also because this form contains a completed square.

Students need to be able to graph quadratic functions and to evaluate them at a given number. They need to be able to transform an equation for a quadratic function from one type to another. For this, they must be able to multiply linear factors, factor quadratic expressions, and complete the square. They need to match graphs to equations and vice-versa.

Assessment Resources

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N07: Building Functions	A06: Sidewalk Patterns	E07: Skeleton Tower
	A07: Functions	E13: Sidewalk Stones

A6 Quadratic Equations
26 Days

In this unit, students use algebra to transform quadratic expressions and equations and to solve quadratic equations in one variable. This includes factoring quadratic expressions as well as using the quadratic formula and completing the square to solve quadratic equations. Students extend their number sense by investigating complex numbers within the context of quadratic equations with complex solutions. Finally, they build on their earlier work with linear systems by using algebra and graphing to solve systems consisting of a linear equation and a quadratic equation in two variables.

Seeing Structure in Expressions A-SSE
3a. Factor a quadratic expression to reveal the zeros of the function it defines.
3b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

Reasoning with Equations and Inequalities A-REI
4 Solve quadratic equations in one variable
4a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.
4b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Formative Assessment Lessons

Solving Quadratic Equations: *Cutting Corners*

In this lesson, students need to figure out the distance that the back wheel of a bus cuts into a bike lane as it turns with its front wheels going over the outside edge of the bike lane. The information given is the distance between the front and back wheels, and the radius of the outside of the bike lane.

Students need to make a drawing modeling the situation, which leads to a right triangle. They must use the Pythagorean theorem in order to obtain a quadratic equation involving the given information and the distance they need to find. Finally, they must solve this quadratic equation. They can complete the square or use the quadratic formula, and they can also make a linear change of variables leading to an easier quadratic equation. Once they solve this equation, they must interpret the algebraic solutions in the given context in order to keep the correct value and discard the other, which is larger than the outside bike lane radius and hence not what they are looking for.

Next, students need to find how far the front wheel of the bus must be from the outside edge of the bike lane in order for the back wheel not to cut into the lane. This requires a similar procedure, involving the same ideas as above, but the variables change slightly. Students are also asked to analyze, compare, and evaluate four solution attempts provided to them as sample student work.

Teaching and Assessment Resources

http://map.mathshell.org/materials/index.php

http://www.MathEducationPage.org/

http://illustrativemathematics.org/

S1 Statistics
30 Days

This statistics unit contains two main ideas: interpreting data and modeling data. When working with data, there is always the temptation to stop at calculating measures of center and spread. A key focus of these standards is *interpreting* these measures in the context of the data sets. Comparisons between graphs of multiple data sets should also be emphasized. These comparisons should be based on descriptions of shape, center, spread, and extreme values. Students need to examine data using multiple representations, including various graphs, lists, and tables.

The other focus of this unit is modeling data. Linear models are emphasized, but quadratic and exponential models are mentioned. While regression is not formally discussed in this unit, foundational understandings are developed to ease this process in the future. Not only should linear models be approximated for appropriate data sets, but they should also be interpreted in context of the data set. Effort should be spent developing a conceptual understanding of correlation and causation. Students should be well aware that correlation does not imply causation.

Interpreting Categorical and Quantitative Data S-ID

1. Represent data with plots on the real number line (dot plots, histograms, and box plots).
2. Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
3. Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
4. Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.
5. Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data.
6. Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

6a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*

6b. Informally assess the fit of a function by plotting and analyzing residuals.

6c. Fit a linear function for a scatter plot that suggests a linear association.

7. Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
8. Compute (using technology) and interpret the correlation coefficient of a linear fit.
9. Distinguish between correlation and causation.

Formative Assessment Lessons

Representing Data 1: Using Frequency Graphs

This lesson and *Representing Data 2: Using Box Plots* are very similar, and they share some materials for students to work on. The goal is for students to interpret and analyze data about the frequency of an event using frequency graphs and verbal descriptions, and connecting the two representations.

Although the data are finite and hence discrete (about a group of students and their monthly cell phone spending or their scores on a test), students are presented with continuous frequency graphs of the situation, and they need to understand how the continuous graph approximates and relates to the real, discrete data. For example, they need to approximate the area under the graph to make deductions about the sample size (number of students) and about the median value. For making these approximations, they may need to triangulate the region under the curve or make a grid of thin rectangles. They are then asked to determine the mode and range of the distribution from the graph. They also need to make inferences about the sampled students or about the test given the shape and range of the frequency graph. Finally, students are asked to match a set of graphs to a set of verbal descriptions in the context of a test.

In order to succeed in this lesson, students must have an understanding of mean, median, mode and range, and be able to estimate them from frequency distributions.

Representing Data 2: Using Box Plots

This lesson goes hand in hand with *Representing Data 1: Using Frequency Graphs*. It is also about data analysis and interpretation of data. But whereas in *Representing Data 1: Using Frequency Graphs* students are asked to match frequency graphs to verbal descriptions, in this lesson they have to match the same frequency graphs to their corresponding box plots.

Students need to understand what information is provided, as well as what information is *not* provided (for example mean, mode, sample size, distribution within each quartile) by a box plot. They also need to understand that there is a unique box plot that corresponds to any given frequency graph, but there are many frequency graphs that would produce any given box plot. Students are asked to draw a box plot corresponding to a given frequency graph, and to draw more than one frequency graph with varying features that all correspond to the same given box plot. They are also asked to interpret the box plots and graphs in the context of the data (a group of students who took a test). Again, students need to be able to estimate and to compare areas under the frequency graph in order to approximate the quartiles for the corresponding box plot.

Interpreting Statistics: A Case of Muddying the Waters

This lesson is about interpreting data. Students are given, and asked to act out, complex scenarios resembling real-world situations, where there is a dispute to be resolved in court.

Data are presented to them in several forms, including diagrams, graphs, surveys, tables, pie charts, and scatter plots. Students need to evaluate the claims made on both sides of the arguments, and they need to think critically about the inferences or conclusions drawn from the data by the different parties. In order to do so, they must attend to several issues that can result in misleading or manipulative data, like bias of samples, bias of questions, and scaling of graphs. They need to consider the importance of the size of samples and surveys and whether or not they can be biased because of the way they were collected. Students also need to evaluate claims about causal relationships based on the data. They need to consider plausible alternative causes not shown by the data. Overall, the lesson demands analytical and critical thinking, as well as communication and debating skills.

Devising a Measure for Correlation

This is a lesson about mathematical modeling in the context of statistical correlation between two variables. Students are asked to describe three different scatter graphs showing several data points. One graph shows a strong correlation, another shows a weak correlation, and the third shows no correlation. They are also presented with a scatter graph with fewer points, and asked to provide additional data points that would increase or decrease the correlation between the variables.

Students are asked to carefully analyze four different models for obtaining a correlation coefficient: one in the first assigned task, and three other methods provided as sample student work, after they have attempted to come up with models of their own. They need to evaluate and compare all the models (those provided as well as their own and each other's). The methods provided include considering the area enclosed by the convex hull of the data points, looking at the vertical distance between data points and an estimated line of best fit, splitting the graph into four equal rectangular regions and counting the number of data points in them (two regions correspond to strong correlation and the other two to weak correlation), and comparing the *x*- and *y*- ranking (order) of the data points.

Students need to consider, for each model, what range of values the method can yield, examples of data that may not be well accommodated by the model, sensitivity to outliers, and the result of changing scales on the axes. Students are not required to know a formula for the correlation coefficient, but they need to have a common-sense understanding of its meaning and they need to be able to describe the correlation by seeing the scatter graphs.

Teaching and Assessment Resources

"The Hand Squeeze" – an activity for collecting data that can be modeled with a linear model. http://math.rice.edu/~lanius/Algebra/hndsq.html

NCTM Advanced Data Grapher—an applet for graphing data. http://illuminations.nctm.org/ActivityDetail.aspx?ID=220

NCTM Histogram Plotter—an applet for creating histograms. http://illuminations.nctm.org/ActivityDetail.aspx?ID=78

NCTM Line of Best Fit estimator—an applet for estimating a line of best fit for a set of bivariate data. http://illuminations.nctm.org/ActivityDetail.aspx?ID=146

NCTM Box and Whisker/Mean and Median applet—an applet that compares box plots with data sets. http://illuminations.nctm.org/ActivityDetail.aspx?ID=160

The following tasks can be found at http://map.mathshell.org/materials/tasks.php :

Novice Tasks	Apprentice Tasks	Expert Tasks
N11: Interpreting Categorical and Quantitative Data	A09: sugar Prices	